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METRIC PROPERTIES OF MULTIDIMENSIONAL STIMULUS CONTROL.

CROSS, DAVID V.
KUM37771 UNIVERSITY OF MICHIGAN, ANN ARBOR

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THIS STUDY FOCUSED ON THE GENERAL PROBLEM OF HOW THE TOTAL EFFECT OF A MULTIDIMENSIONAL STIMULUS IS COMPOUNDED FROM THE SIMPLE EFFECT OF ITS SEPARATE COMPONENTS IN THE CONTEXT OF STIMULUS GENERALIZATION. FROM A THEORETICAL LEVEL, THE PROBLEM WAS STUDIED IN TERMS OF THE GEOMETRY OF MINKOWSKI. AN EXPERIMENT WAS CARRIED OUT WITH 20 SUBJECTS WHO WERE TRAINED ON SUCCESSIVE DISCRIMINATION INVOLVING 3 FREQUENCY MODULATED SINUSOIDAL TONES THAT DIFFERED IN CENTER FREQUENCY AND/OR THE RATE OF MODULATION. THE PATTERN OF REINFORCEMENT FOR THE EMISSION OF EITHER OF TWO MUTUALLY EXCLUSIVE RESPONSES WAS CONTINGENT ON THE PRESENCE OF ONE OR TWO VISUAL CUES. IN THIS WAY, THE STIMULUS PROPERTY ATTENDED WAS BROUGHT UNDER EXPERIMENTAL CONTROL. THE STRUCTURE INDUCED ON THE EXTENDED STIMULUS SET WAS EXAMINED USING GENERALIZATION TESTING PROCEDURES WITH THE TWO VISUAL CUES PRESENTED SINGLY FOR 10 SUBJECTS, AND WITH CUES PRESENTED IN COMBINATION FOR THE OTHER 10 SUBJECTS. MEASURES OF THE RESPONSE PROBABILITY AND RESPONSE LATENCY WERE OBTAINED. MEASURES OF DISCRIMINATION LATENCY WERE RELATED TO MEASURES OF RESPONSE PROBABILITY. IN THE CHOICE OF A MODE OF EXPRESSION FOR RESPONSE LATENCY, THE MOST REASONABLE CANDIDATES WERE THE RECIPROCAL AND LOGARITHMIC TRANSFORMATIONS. (JC)

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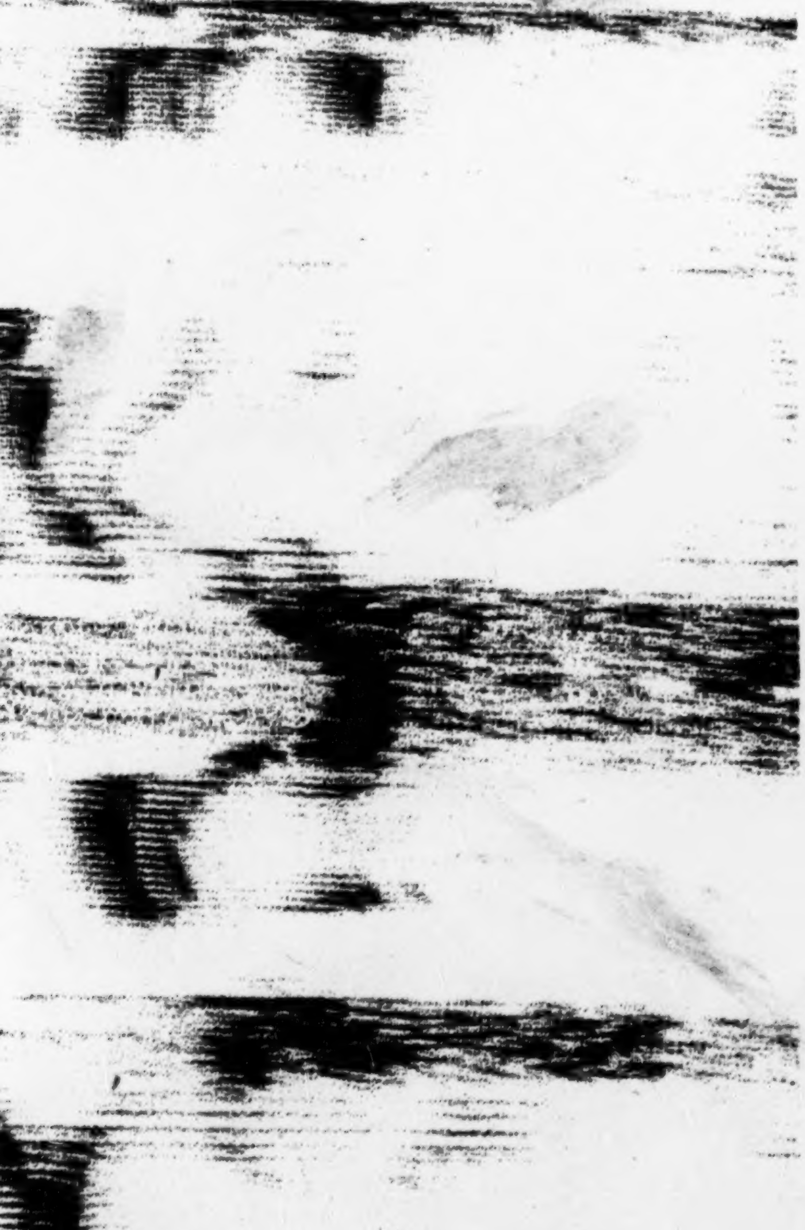

Studies in Language and Language Behavior

METRIC PROPERTIES
OF
MULTIDIMENSIONAL STIMULUS CONTROL

CENTER FOR RESEARCH ON LANGUAGE AND LANGUAGE BEHAVIOR

METRIC PROPERTIES OF MULTIDIMENSIONAL STIMULUS CONTROL

by
David Vernon Cross

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Doctoral Committee:

Professor Clyde H. Coombs, Chairman
Professor Frank Harary
Associate Professor Harlan L. Lane
Assistant Professor William C. Stebbins

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TABLE OF CONTENTS

Page

Acknowledgements ii

List of Tables v

List of Figures vi

Chapter

I. Multidimensional Stimulus Control of the Discriminative Response in Experimental Conditioning and Psychophysics 1

Aspects of Stimulus Control: Generalization and Discrimination 1

Rules of Combination in Multidimensional Stimulus Control 4

Experimental Conditioning with Compound Stimuli 5

Metric Analyses in Multidimensional Psychophysics 8

Metrization of Multidimensional Stimulus Control 13

Metric Implications of Equal-Level Surfaces 16

Metric Treatment of Nonlinear Stimulus-Response Relations 20

Metric Models for Multidimensional Stimulus Generalization 25

A General Minkowski Model for Multidimensional Stimulus Generalization 31

Empirical Evidence of the Metric Properties of Multidimensional Stimulus Control 36

Summary 43

II. Attentional Factors in Multidimensional Stimulus Control 46

Experimental Evidence for Attentional Factors in Stimulus Control 47

Attention and the Metric Structure of Pooled Data 54

Interaction of Continua and Effective Dimensions of Control 58

Summary 61

TABLE OF CONTENTS (continued)

Chapter Page

III. An Experimental Comparison of Alternative Measures of Stimulus Control under Conditions of Selective Attention of Single Stimulus Properties.............................. 62

Introduction.. 62

Method.. 63

Results... 67

Discussion.. 89

IV. Summary and Conclusions.. 94

References... 97

LIST OF TABLES

Table Page

1. Analysis of variance of auditory generalization measured in terms of total R_1 frequency: Group I........................ 70

2. Analysis of variance of mean log response latency to stimuli varying in center frequency and modulation rate under amber and blue light conditions: Group I.................. 75

3. Summary of results for Group II.................................. 84

LIST OF FIGURES

Figure Page

1. Level contours arising from nonlinear stimulus-response relations.. 22
2. Possible forms of the generalization surface in two dimensions... 26
3. Euclidean graphs of Minkowski "unit-circles"...................... 33
4. Auditory generalization under visual control...................... 69
5. Response latency under the amber light condition.................. 72
6. Response latency under the blue light condition................... 73
7. Group I measures of response probability and latency pooled across light conditions.. 77
8. Response probability and latency under the combined light condition: Group II... 83
9. Group II measures of response probability and latency............ 86

CHAPTER I

Multidimensional Stimulus Control of the Discriminative Response in Experimental Conditioning and Psychophysics

Aspects of Stimulus Control: Generalization and Discrimination

An animal that learns to emit a particular response in the presence of a specific combination of stimulus properties will emit the same response when one or more of these properties are altered or varied along underlying continua. Thus, a rat that learns to avoid electric shock by jumping over a barrier upon presentation of a 750 cps tone will also jump when a 2,500 cps tone is presented, even though the latter tone had never been paired with shock. Or the key-pecking behavior of a pigeon that is reinforced only when the response key is trans-illuminated by a red light, and is extinguished when the key is dark, may also be evoked when a green light illuminates the key. A response is said to be under stimulus control when its rate or probability of emission is observed to vary with the presence and absence of a stimulus. As exemplified above, when stimulus control is established experimentally, the fact of control is not restricted to the specific combination of physical stimulus properties initially correlated with reinforced responding, but is diffused over a wide class of specifiable stimuli. This lack of specificity that characterizes the discriminative (response-evoking) function of a stimulus is called stimulus generalization.

Generalization is exhibited in the behavior of the monkey that avoids all tigers, not just the first one with which it had an unpleasant encounter; and of the child who learns to call its father "Daddy" and then calls all men "Daddy." An object viewed under different light conditions or from different angles of perspective may present widely disparate conditions of stimulation yet evoke the same previously learned identification response. In a perpetually changing environment in which stimulus conditions seldom recur in identical form, it is through generalization that an organism's behavior exhibits consistency and stability.

On the other hand, behavior may show remarkable specificity with respect to stimuli. Animals can learn to respond differentially in the presence of very slight stimulus differences. Pigeons can be trained to discriminate monochromatic lights differing by as little as one millimicron in wavelength (Hanson, 1956). The pre-medical student who is initially unable to discern even the gross features of a tissue preparation viewed through a microscope may become, through extensive training, the pathologist who is able to detect the slightest abnormalities in cell structure that indicate the presence of malignancy in a tumor. When differential responding develops in the face of generalization we call it stimulus discrimination. The professional wine-taster can make very fine discriminations; he shows little generalization among various wines. The phonetician discriminates speech sounds which seem identical to (or are generalized by) most untrained listeners.

The operations entering the above definitions of generalization and discrimination serve, at one level of analysis, as dichotomous classifiers of stimuli with respect to some arbitrarily specified behavior. Two different stimuli may evoke either the same behavior (generalization), and thus be classified as equivalent with respect to the class of responses they control, or they may evoke different behavior (discrimination) and be assigned to distinct classes. The set of all experimentally specified stimulus events are thus partitioned into two classes, with class membership contingent upon the response-evoking (discriminative) function of the stimulus. This analysis by classification is not restricted to the occurrence versus non-occurrence of a single unit of behavior. Consider as an example of more complex classification an experiment in which a human subject is instructed to label by color name each of a large set of Munsell color patches. Since there are fewer commonly-used labels than there are samples in the Munsell system, the subject's behavior will partition the stimulus set into response-specific subsets or classes. Generalization with respect to the verbal response "blue" is evidenced by the large number of different color patches assigned that label, whereas discrimination is evidenced when two stimuli are assigned different labels.

In quantitative analyses of stimulus-response relations generalization and discrimination are viewed not as dichotomous apsects of stimulus control but rather as quantitative variables representing the amount or degree of discriminative control a stimulus exerts over behavior. A stimulus which consistently evokes a particular response is said to have greater control over that response than does a stimulus which evokes the response only part of the time, that is, with smaller probability. Or two different stimuli may evoke the same response but the latency of response (the time lapsing from stimulus onset to occurrence of the response) may be short for one stimulus and long for the other. A pigeon may key-peck at a steady and continuous rate in the presence of one stimulus, whereas key-pecking in the presence of a generalized stimulus may be characterized by breaks in rate, that is, by periods of responding interspersed with not responding. Measures of response strength derived from these observations (or from other experimental operations) provide a basis for differentiating stimuli which are otherwise given the same classification. The stimulus generalization gradient--defined operationally in relation to changes in the discriminative stimulus as a succession of decrements in response probability or in some other measure of response strength--identifies the class of stimuli that are effective in evoking a particular response and, in addition, defines the degree of membership for each stimulus in that class.

From a strictly operational point of view generalization and discrimination may be considered alternative and opposite ways of describing the same set of stimulus-response relations (a view either explicitly or implicitly held by numerous investigators: e.g., Lashley & Wade, 1946; Gibson, 1959; Mednick and Freedman, 1960; Prokasy & Hall, 1963; Brown, 1965). "The experimentally conditioned [stimulus], and any other stimulus on a physical-property continuum with it, are said to generalize to the extent that they overlap in capacity to evoke R; to the extent that they differ in evocative capacity, they are said to be discriminated." (Schoenfeld & Cumming, 1963, p. 225) If two stimuli set the occasion for different levels of responding (represented by different points on the generalization gradient) they are exerting differential control over behavior (discrimination) and the difference in their response measures may be interpreted either as a measure of discrimination or as a measure of generalization difference.

Rules of Combination in Multidimensional Stimulus Control

Stimulus generalization and discrimination have a long and interesting history of experimental study and they have been investigated from many viewpoints, but little attention has been directed toward the problem of how generalization (or discrimination) occurs in the presence of stimuli that vary simultaneously on two or more dimensions as compared with generalization (or discrimination) occurring in the presence of stimuli what vary along single dimensions only. What rule of combination predicts the behavioral control exerted by a multidimensional stimulus as a function of the control exerted separately by each of its component properties? This problem pervades all levels of behavior, simple and complex, animal and human, since the stimuli which set the occasion for differential responding outside of the laboratory characteristically differ on two or more dimensions and may even involve different sensory modalities. How does the behavior evoked in these situations compare with discriminative behavior that is under the control of unidimensional stimuli? What are the rules that govern how unidimensional stimulus effects combine to produce multidimensional stimulus control? These questions may be raised with regard to a variety of experimental problems: whether we are concerned with the course of development of stimulus control, as in discrimination-learning experiments; with the specifity of stimulus control as in generalization experiments; with the relation between specific properties of stimuli and presumptive measures of sensory magnitude, as in psychophysical experiments; or with the general question considered in multidimensional scaling theory of how different stimulus attributes combine to affect the total similarity of perceived objects.

Experimental Conditioning with Compound Stimuli. Research in the area of experimental conditioning has been concerned with the operations that establish a stimulus-response relation where none, or a different one, existed before, usually with emphasis on the course of development or the specificity of control as a function of other variables (e.g., drive level, reinforcement schedule, etc.). These experiments include studies of stimulus generalization, in which a response is brought under the control of a particular stimulus complex and then tested in the presence of other stimuli sampled from a common physical-property continuum. Included also are conditioning experiments and discrimination-learning experiments in which the dependent variable is either the different number of stimulus presentations or trials required to establish control, or the degree of control established as measured by differential amounts of conditioning or the number of correct discriminations obtained. Few of these studies have investigated conditioning as a function of the dimensionality of the stimulus variable.

The studies referred to here were not concerned with the specific rule of combination that predicts multidimensional stimulus control but rather with the more elementary question of whether stimuli differing on two or more dimensions exercise more behavioral control than do stimuli which differ on one dimension only. In spite of the quantitative limitations, the results of these studies provide a basis for some rather broad inferences about rules of combination in the multidimensional case. For example, Miller (1939) found that a combined auditory and visual stimulus (a drop in pitch of a continuous tone, and the movement of a pointer), paired with electric shock to the cheek, resulted in faster and stronger conditioning of an eyelid response in humans than did either stimulus property used alone. Similarly, Eninger (1952) found that rats formed a discrimination much more rapidly when auditory and visual cues were combined (tone vs. no-tone, and white vs. black) than when either of these sets of cues was presented alone. In a generalization study, Fink & Patton (1953) controlled certain visual, auditory, and tactile characteristics of the stimulus situation in which rats learned to drink water from a tube. After establishing a stable baseline of drinking rate, the investigators altered one or more of the stimulus properties under their control. They found that any change led to a decrement in drinking: light changes were most effective, sound changes were moderately effective, and tactual changes were least effective in modifying the strength of the learned drinking response. More important, however, they found that a change in any two stimulus properties resulted in a greater response decrement than a change in any single property; when all three stimuli were changed, maximum response decrement occurred. White (1958) trained children to pull a handle to get marbles, using a card of a certain hue and saturation (Munsell 10GY 8/6) as a discriminative stimulus. Then, for different groups of subjects, he changed the hue of the stimulus card, its saturation, or both, in a series of unreinforced test trials. There was greater generalization decrement to test stimuli which differed from the training stimulus in both dimensions than to those which differed in either of the dimensions alone. That is, the properties of hue and saturation combined to exert more control over differential responding than either property exerted alone. A generalization suggested by these results is that every stimulus property that happens to be correlated with the contingencies of reinforcement requiring differential responding acquires some degree of stimulus control; when stimulus properties are presented in combination, their effects combine in some manner to exert an even greater degree of control.

On the other hand, it has been found, in some discrimination experiments, that stimuli differing in two properties are discriminated with no greater ease or efficiency than stimuli differing in the single one of these properties which is most readily discriminated. In other words, the degree of behavioral control exerted by the several properties of a compound stimulus is no greater than that exercised by the component property which, by itself, is most effective. Harlow (1945) trained monkeys to respond discriminatively to stimuli differing in either color, form, or both. Stimuli differing in both color and form, or in color alone, were discriminated equally well, although both were discriminated more readily than form. This result was also found by Warren (1953) who tested monkeys on discrimination problems involving stimuli which differed in color, form, size or in the four combinations of these properties taken two or three at a time. Stimuli which differed in color and some other property were little, if any, more discriminable than those differing in color alone. In a later experiment, Warren (1954) trained monkeys to discriminate pairs of stimuli which differed simultaneously in color, form, and size. After establishing stimulus control, the number of discriminative cues was reduced by eliminating stimulus differences in one or two dimensions. In each instance, performance scores were depressed. Discriminative control was almost completely lost when color differences were eliminated, regardless of which other property or combination of properties remained. Some control was lost when form and size differences were both eliminated, and the smallest decrements in performance occurred when only form or size differences were eliminated, size being the least effective.

Thus, results of some experimental studies indicate that behavioral control is distributed over the several dimensions of a complex discriminative stimulus, while the results of others suggest that a single dimension overshadows the rest and is dominant in controlling differential responding. In other words, the combination of stimulus properties results not in a summation of their separate effects but in an effect equal to that of the property which, alone, is most effective. It is evident that no one combination rule can account for these contradictory findings.

Metric Analyses in Multidimensional Psychophysics. Traditionally the study of "similarity" relations imposed by behavioral measures on pairs of stimuli that differ with respect to several physical properties at once has been called "multidimensional psychophysics." Analysis of experimental data obtained in this line of research has almost universally been guided by the assumption that the collection of N stimuli employed in an experiment can be represented by a spatial configuration of "points" in a multidimensional Euclidean space. The configuration is determined by the $\binom{N}{2}$ inter-point distances which are derived from behavioral measures obtained on each stimulus pair. The experimental data can be displayed in an N x N matrix, with each row and corresponding column associated with a particular stimulus and each cell containing a response measure, P_{ij} (pertaining to the row stimulus S_i and the column stimulus S_j), which depends on the experiment and behavior sampled. These are called proximity measures by Coombs (1960) because they are interpreted as indicating how closely stimuli S_i and S_j are related. In a generalization experiment P_{ij} may represent the amount of generalization between a training stimulus S_i and a test stimulus S_j, as measured by response probability, latency or rate; or P_{ij} may correspond to the percent "same-different" judgments evoked in a discrimination task; or to the "similarity" of S_i and S_j as rated by subjects on a scale from, say, 0 to 10. These measures were later classified as symmetric or conditional proximity data, depending on whether $P_{ij} = P_{ji}$, or $P_{ij} \neq P_{ji}$ (Coombs, 1964a).

Whatever the behavioral measures obtained in a given experiment a function is postulated which transforms them into a set of presumptive distances D_{ij} which may be analyzed by a multidimensional scaling algorithm, first proposed by Young and Householder (1938), and subsequently refined by Torgerson (1952), and by Messick and Abelson (1956). The purpose of the analysis is to determine the effective dimensionality of the space in which the stimuli are assumed to be imbedded and to determine the projections of the stimuli on each of the dimensions involved. The method invokes the law of cosines in order to transform the matrix of distances $[D_{ij}]$ into a matrix $[B_{ij}]$ of scalar products of stimulus vectors, having as their common origin the centroid of the points. The matrix $[B_{ij}]$ is then factor-analyzed in much the same way as correlation coefficients. The coordinates for each stimulus on orthogonal axes in psychological space are then given by their factor loadings.

A study by Shepard (1958) will serve to illustrate this procedure. The study was an identification-learning experiment in which subjects were required to learn a different arbitrarily-specified response to each of nine Munsell colors, varying in brightness (value) and saturation (chroma). In the course of the experiment generalization occurred, that is, the response appropriate to, say, stimulus S_i was also evoked by S_j. Generalization from S_i to S_j, denoted by $g(i,j)$, is represented by the ratio $g(i,j) = n_{ij}/n_{ii}$ where n_{ij} is the number of S_j presentations resulting in the response appropriate to S_i, and n_{ii} is the number of times the response is correctly evoked by S_i. Shepard postulated, after Hull (1943), that $g(i,j)$ is an exponential decay function of the psychological distance between S_i and S_j, that is,

$$g(i,j) = e^{-kD_{ij}} \qquad \text{or} \qquad D_{ij} = -k' \log_e g(i,j) \quad .$$

Thus, the distance between the points, in "psychological space", corresponding to S_i and S_j is given by the logarithm of the corresponding measure of generalization. However, two separate measures of generalization were obtained for each stimulus pair, $g(i,j)$ and $g(j,i)$, and each gave rise to a different estimate of the same distance. So, in order to obtain a symmetric distance-matrix, Shepard simply averaged the two. The resulting distances were then reduced to an orthogonal set of coordinates by the method indicated above. Actually, this is a somewhat simplified account. Shepard also calculated weights for each stimulus, which were intended to correct for response biases and the asymmetries that occurred, and determined a constant which theoretically took account of the essentially random responses occurring during early learning trials. In an attempt to evaluate the goodness-of-fit of the scaling model to the empirical data, Shepard reconstructed response measures from the weights and final factor loadings computed for each stimulus. He reported that approximately 99 percent of the variance in the original response measures was accounted for by reconstruction.

Two important theoretical assumptions guided this analysis. The first concerns the form of the generalization function, and the second concerns the rule that governs how differences along two or more dimensions combine to determine the overall difference between two stimuli. The assumption that psychological space is Euclidean in character requires that the total difference between two stimuli be defined according to the Pythagorean theorem, as the square root of the sum of squared differences in projections upon orthogonal reference axes (dimensions). This is the combination rule adopted explicitly, or implied, by most multidimensional scaling models. On the face of it there appears to be no good reason to believe that this is in fact how stimulus control is distributed over the various dimensions of pscyhological space. There is certainly no conclusive experimental evidence to indicate that this is so. The truth of the matter is that the adoption of the Euclidean combination rule has simply followed the dictates of mathematical convenience. The analytic procedures of most scaling models exploit certain mathematical relations that are valid only in Euclidean spaces.

The methods of multidimensional scaling have been and will continue to be useful techniques for: (1) the construction of coordinate systems descriptive of stimuli of unknown dimensionality or of stimuli on which the physical measurements that can be made are not directly related to the behavior under investigation; or (2), for reducing a large set of correlated measures to a smaller set of orthogonal linear components which account for most of the variance in the original measures. However, the practice of demanding reference to a Euclidean metric for behavioral data that come equipped with their own metric (the combination rule that actually generates the data) imposes constraints upon the analysis that may result in an unwanted misrepresentation of the behavior, in terms of both the combination rule and the dimensionality of the solution. If the selection of a particular metric embedding is to have any heuristic value it should provide not only a convenient multidimensional representation of the stimuli, but also a valid description of the behavior in question, and a basis for important theoretical conclusions and techniques for comparing and testing alternative theories of behavior.

Unfortunately, very few studies have approached the problem of multidimensional data analysis from the point of view of evaluating the metric, or combination rule, that actually generated the data. Several authors have suggested that a non-Euclidean metric may be appropriate (e.g., Attneave, 1950; Coombs, 1951; Galanter, 1956; Torgerson, 1958; Shepard, 1964). Coombs (1964b) has discussed the advantages of exploring alternative metric representations of behavioral data. In his own words "one of the most desirable consequences of developing alternative models and their algorithms for data analysis is that *their existence destroys any naive complacency with any one model and leads to a search for ways of testing and comparing alternative theories*." (p. 206, italics in original) The point is that the algorithms used in multivariate data analysis and the metric embeddings of multidimensional scaling models are, by their nature, theories about behavior. They are theories about how unidimensionallyscaled stimulus-effects combine to determine the behavior evoked in multidimensional situations. There is not necessarily just one combination rule to which all behavior conforms, but rather alternative rules which may correspond to alternative theories about behavior, or to alternative behaviors, or to behavior under different circumstances, or under the control of different kinds of stimuli.

Metrization of Multidimensional Stimulus Control

If the properties of discriminative behavior under multidimensional stimulus control can be suitably represented by the mathematical properties of *metric spaces*, they can then be analyzed in terms of the mathematical procedures of the geometries associated with these spaces. This would not only assist us in the practical business of dealing with multidimensional stimulus variables, but also provide a formal basis for characterizing and contrasting alternative theories or hypotheses about behavior.

The basic concept underlying the notion of a metric space is that of the *distance* between two points. The properties of distance will depend to some extent on the space considered; but certain basic properties are definitive and assumed always to hold. These are: (1) the distance between any two points is non-negative and only the distance of a point from itself is zero; (2) the distance between any two points is symmetric, that is, the distance from the first point to the second is the same as that of the second to the first; and (3) for any three points the distance between one pair is no greater than the sum of the other two distances, a condition called triangle inequality.

These assumptions can be stated a little more exactly by formulae. A set S is *metrizable* if and only if to each pair x, y of its elements we can associate a non-negative real number $d(x, y)$, called their distance, which satisfies the conditions: (1) $d(x, y) = 0$, if, and only if, $x = y$; (2) $d(x, y) = d(y, x)$; (3) $d(x, z) \leqq d(x, y) + d(y, z)$, for all x, y, z, in S. The function $d(x, y)$ is said to be a metric for S.

The ordinary physical space of three dimensions is a metric space with Euclidean distance. The above conditions are quite general, however, and are also satisfied by the following example. Take a set of color patches and for each pair x, y define the distance $d(x, y) = 0$, if both members of the pair are labelled with the same color name, and define $d(x, y) = 1$, if labelled differently. The function $d(x, y)$ could serve as a metric for the color domain, but it does not provide a basis for a very interesting or especially informative model for color perception.

The metric exioms clarify the conditions under which measures of stimulus control may be treated as measures of distance. However, in the collection of single stimulus data, as in generalization and some discrimination training paradigms, the task of empirically verifying each of these conditions for all stimulus pairs can be formidable. Consider a stimulus generalization study involving N stimuli and yielding measures of generalization decrement, defined as the difference in the response measure produced by the training stimulus and each test stimulus. The question, here, is whether these quantities can serve as measures of distance between stimuli. Direct tests of the symmetry and triangle inequality properties of distance requires N replications of the study, each time with a different training stimulus, in order to fill in the N^2 cells of the data matrix. This undertaking can be expensive in terms of research time and in the consequences of failure to control all variables affecting stimulus control in the separate replications.

On the other hand, if the effective dimensions of stimulus control are presumed to be known (or at least referable to the physical dimensions of the stimulus), then the question of whether measures of generalization decrement define a metric for the stimulus set can be answered by evaluating empirically the combination rule that relates the amount of generalization decrement for each unidimensional stimulus change to the total decrement obtained when all changes are effected simultaneously.

If we denote by g(S) the generalization measure corresponding to the discriminative stimulus, S, that maximally evokes R during generalization testing, and we let $g(S_{\Delta x_i})$ represent generalization to a test stimulus differing from S along dimension x_i only, by an amount Δx_i, then the generalization decrement produced by Δx_i is defined as

$$\overline{g}(S_{\Delta x_i}) = g(S) - g(S_{\Delta x_i})$$

and the combination rule we seek can be expressed as the function, F, that relates these measures to the total generalization decrement produced by the stimulus differing from S simultaneously with regard to all Δx_i ($i = 1, \ldots, n$); that is

$$\overline{g}(S_{\Delta x_1}, \ldots, {}_{\Delta x_n}) = F\left[\overline{g}(S_{\Delta x_1}), \ldots, \overline{g}(S_{\Delta x_n})\right]$$

In the space, P, defined by the physical coordinates of the stimuli, the configuration of points corresponding to a given level of $\overline{g}$ (with reference to the fixed point, S) depends on the function F. Different combination rules result in different configurations or "level-surfaces" and the shape of these surfaces determines the metric, if any, imposed on the stimulus space by $\overline{g}$. In the Euclidean space of two dimensions, for example, systems of points that are equally distant from a fixed point lie on concentric circles with the fixed point as center; the circle $x^2 + y^2 = 1$ being, so to speak, the standard unit circle.

But the Euclidean space is only a special case of a general class of metric spaces for which consistent geometries have been developed and which may serve to characterize the properties of behavior under multidimensional stimulus control. These are called Minkowski spaces and their geometries require only that the level-surfaces in P be of the same centrally symmetric and convex shape. These conditions are discussed in the following section.

Metric Implications of Equal-level Surfaces. Let $P_1, \ldots, P_n$, respectively, denote each of the n physical-property continua or dimensions (e.g., wavelength, intensity, size, etc.) called upon to operationally characterize the stimuli involved in, say, a generalization experiment. Each stimulus can be represented by a point in a physically-defined space P, where

$$P \equiv P_1 \times \ldots \times P_n \equiv \{p_1, \ldots, p_n): \; p_i \in P_i \; (i = 1, \ldots, n)\} \quad .$$

The space P may be viewed as an n-dimensional Cartesian space with orthogonal reference axes P_i $(i = 1, \ldots, n)$ intersecting at a common point θ which is taken as the origin of the space. If we take as θ the point representing the particular combination of physical aspects that specifies the initial training stimulus in the generalization experiment, then each test stimulus p may be represented by the ordered n-tuple $p = (p_1, \ldots, p_n)$, the components of which pertain to the signed physical differences between p and θ on each of the n dimensions of P; the quantities p_i $(i = 1, \ldots, n)$ are the relative coordinates of p with regard to θ.

Each P_i is a unidimensional metric space with the metric defined by $d_i(p_i, q_i) = |p_i - q_i|$ for every pair of points $p_i, q_i \in P_i$; in other words, $d_i(p_i, q_i)$ is the physical separation of $p, q \in P$ on the single dimension P_i. Since the Cartesian product of n metric spaces is also a metric space, it is possible to define a function $d: P \times P \rightarrow R$ (the Euclidean distance function, for example) which assigns to each pair of points $p, q \in P$ a real number $r \in R$ (the set of real numbers) interpreted abstractly as the physical distance between the stimuli denoted by p and q. We might ask the following question: under what conditions is it possible to construct a metric for P in such a way that the distance $d(\theta, p)$, defined for fixed θ and any p in P, is equivalent to the difference in behavioral control (e.g., generalization decrement) exerted by the two stimuli represented by θ and p.

Let $\bar{g}(\theta, p)$, or simply $\bar{g}(p)$, represent the difference in behavioral control exerted by θ and p in stimulus generalization, that is, $\bar{g}(p) = g(\theta) - g(p)$, where g is the generalization function which we assume has its maximum valuation at θ. Consider the point set $U = \{p \in P: \bar{g}(p) \leq u\}$, containing only those points in P that produce a generalization decrement relative to θ less than or equal to some arbitrary positive value u. The measure of generalization decrement $\bar{g}(p)$ can be taken as a Minkowski metric for P if and only if:

(i) the point set U is symmetric with respect to θ,

(ii) the point set U is convex, and

(iii) the boundary of U is homothetic to the boundary of every point set generated by a different valuation of the parameter u.

The condition of symmetry simply requires that, if the point p belongs to U, then the point $-p = (-p_1, \ldots, -p_n)$ also belongs to U. The convexity condition requires that for each pair of points $p, q \in U$ the line segment joining them is entirely in U; in other words, each point $r = p + (1 - \lambda)q = (p_1 + (1 - \lambda)q_1, \ldots, p_n + (1 - \lambda)q_n)$, $0 \leq \lambda \leq 1$, is also in U.

The boundary of U, which we will denote by U', is the set of all points p for which $\overline{g}(p) = u$. If (i) and (ii) hold, then this set constitutes a centrally symmetric, convex surface in P. Condition (iii) requires that all such surfaces in P be of the same shape, that is, encased one into the other and produced through dilations or contractions in the ratio u:1, where u is a positive real number.

For u = 1 the set U' is called the standard unit surface for P (the set of all points that are unit distant from θ), and it enters the definition of a distance function for P in the following way: for any two distinct points p, q ε P, let the ray with initial point θ, which is parallel to the directed line segment $\overrightarrow{pq}$ (and with the same sense), meet the surface U' at point r. The Minkowski distance of the points p and q, denoted m(p, q), is defined in terms of the Euclidean metric associated with P in the following way:

$$m(p, q) = \frac{e(p, q)}{e(\theta, r)}$$

where e(p, q) and e(θ, r) are the Euclidean distances in P of the points p, q and θ, r respectively. If p = q we put m(p, q) = 0.

If $\overline{g}(p) = t$, then $t = m(\theta, p) = e(\theta, p)/e(\theta, p')$, where p' is the intersection of the surface U' with the ray [θ, p), and t is the ratio of similitude of a homothetic transformation of P with center θ which carries U' into a surface passing through the point p.

For the points on any Euclidean line L in P, the change to Minkowski metric is merely a change of scale; the distances m(p, q) and e(p, q) are proportional on each line L.

A Minkowski metric in P is invariant under translations and central reflections of the coordinate axes of P. It is not, however, invariant under rotations of the axes about the origin. The Euclidean metric is the only specialization of Minkowski distance that has rotation invariance.

Two Minkowski metrics $m_a(p, q)$ and $m_b(p, q)$ in P are isometric if and only if an affinity exists which maps the unit surface U'_a: $m_a(\theta, p) = 1$ on the surface U'_b: $m_b(\theta, p) = 1$. Special cases of such affine transformations are rotations of the coordinate system, homothetic transformations (a uniform stretching or shrinking of the coordinate axes) and simple elongations and compressions of single axes, sometimes called one-dimensional strains. A Minkowski metric in P is isometric to a Euclidean metrization of P' (where P' results from an affine transformation on the points of P) if and only if the unit surface of P is an ellipsoid.

These properties of Minkowski spaces are discussed in greater detail by Hancock (1939) and Busemann (1950, 1955).

It is readily seen that if $\overline{g}(p)$ represents a Minkowski metric for P, then $\overline{g}$ must be a positive definite, convex and homogeneous function of the first order in its n variables $p_1, \ldots, p_n$. That is:

(1) $\overline{g}(p) > 0$ and only $\overline{g}(\theta) = 0$

(2) $\overline{g}(tp) \equiv \overline{g}(tp_1, \ldots, tp_n) = |t|\ \overline{g}(p)$ for any real number t

(3) $\overline{g}(p + q) \equiv \overline{g}(p_1 + q_1, \ldots, p_n + q_n) \leq \overline{g}(p) + \overline{g}(q)$.

In this form the Minkowski distance of p and q in P is defined as $m(p, q) = \overline{g}(p - q) = \overline{g}(p + (-1)q) \equiv \overline{g}(p_1 - q_1, \ldots, p_n - q_n)$.

Conversely, any combination rule or function F that is defined in P and has these properties is a Minkowski metric. It is evident that the metric exioms stated previously are satisfied by the function $F(p - q)$. The symmetry of distance follows from (2) by setting $t = -1$. From the convexity property (3) the law of triangle inequality follows, for if we set $p' = p - r$ and $q' = r - q$ then

$$F(p' + q') \leq F(p') + F(q')$$

$$F(p - q) \leq F(p - r) + F(r - q)$$

or $$m(p, q) \leq m(p, r) + m(r, q) .$$

Treatment of Nonlinear Stimulus-Response Relations. Shepard (1964) was the first to propose that the combination rules governing multidimensional stimulus control can be determined empirically by investigating the shape of the surfaces in P, corresponding to prescribed levels of control:

> With respect to the physically defined space, a knowledge of these rules is equivalent to a knowledge of the shape of the locus of all stimuli that have any prescribed degree of similarity to any prescribed standard stimulus ... the question of this shape turns out to be equivalent to the question of just what metric is appropriate for the psychological space. (p. 56)

Shepard points out that if psychological space is Euclidean in nature, then (providing the unidimensional psychological scales involved are affine transforms of the corresponding physical continua) the isosimilarity contours in physically-defined space of two dimensions must be affinely equivalent to circles, that is elliptical in shape. If these contours are found to have some non-elliptical form, then psychological space cannot be Euclidean.

However, if generalization decrement is not symmetrically linear with changes in the physical stimulus on each physical property continuum then neither a Euclidean nor any other Minkowskian metrization of P is possible. Unfortunately, this appears to be the rule rather than the exception in most investigations of stimulus control. Where intensive physical continua are involved, it is notoriously the case that discrimination is not an invariant function of the physical separation of two stimuli, but depends on their location on the continuum. Collaterally, the generalization decrement produced by an increase in stimulus intensity does not invariably match that produced by an equivalent decrease in intensity. Furthermore, the generalization decrement produced by a stimulus difference of two units is not invariably twice that produced by a stimulus difference of one unit. These outcomes obviously violate the homogeneity property of metric functions, and in some instances may lead to a violation of convexity (hence triangle inequality) as well. However, nonlinearity of stimulus-response relations does not preclude the possibility of mapping P into a space that _is_ metrizable in terms of stimulus-response relations, and which is Minkowskian.

Special cases arise in the metric treatment of multidimensional data in which transformations of the stimulus variables or transformation of the response variable (or both) may be required in order to transform the coordinates of P into the coordinates of a metric space P' (defined as psychological space), wherein these variables may enter relatively simple and psychologically meaningful relations, and wherein a combination rule may obtain that has generality as a principle of behavior. The cases follow:

1. Choice of stimulus scale. The space P may not be metrizable because of nonlinear deformations of the _equal-level surfaces_ of P in directions parallel to one or more reference axes. See, for example, the two-dimensional _equal-level contours_ plotted in Fig. 1(a). These contours are neither centrally symmetric, homothetic, nor convex. This outcome may occur if generalization gradients are symmetrically linear when the stimuli are "appropriately" scaled in logarithmic units, and a simple "additive" combination rule holds for generalization decrement in two dimensions. In this instance a Fechnerian (logarithmic) transformation on both axes of P maps P into P' that has level contours like those depicted in Fig. 1(d).

The empirical operations involved in the measurement and control of physical stimuli result in quantities or magnitudes, such as sound intensity, frequency, wavelength, etc., which generally enter very complex relations with behavior; on the other hand, relations defined in terms

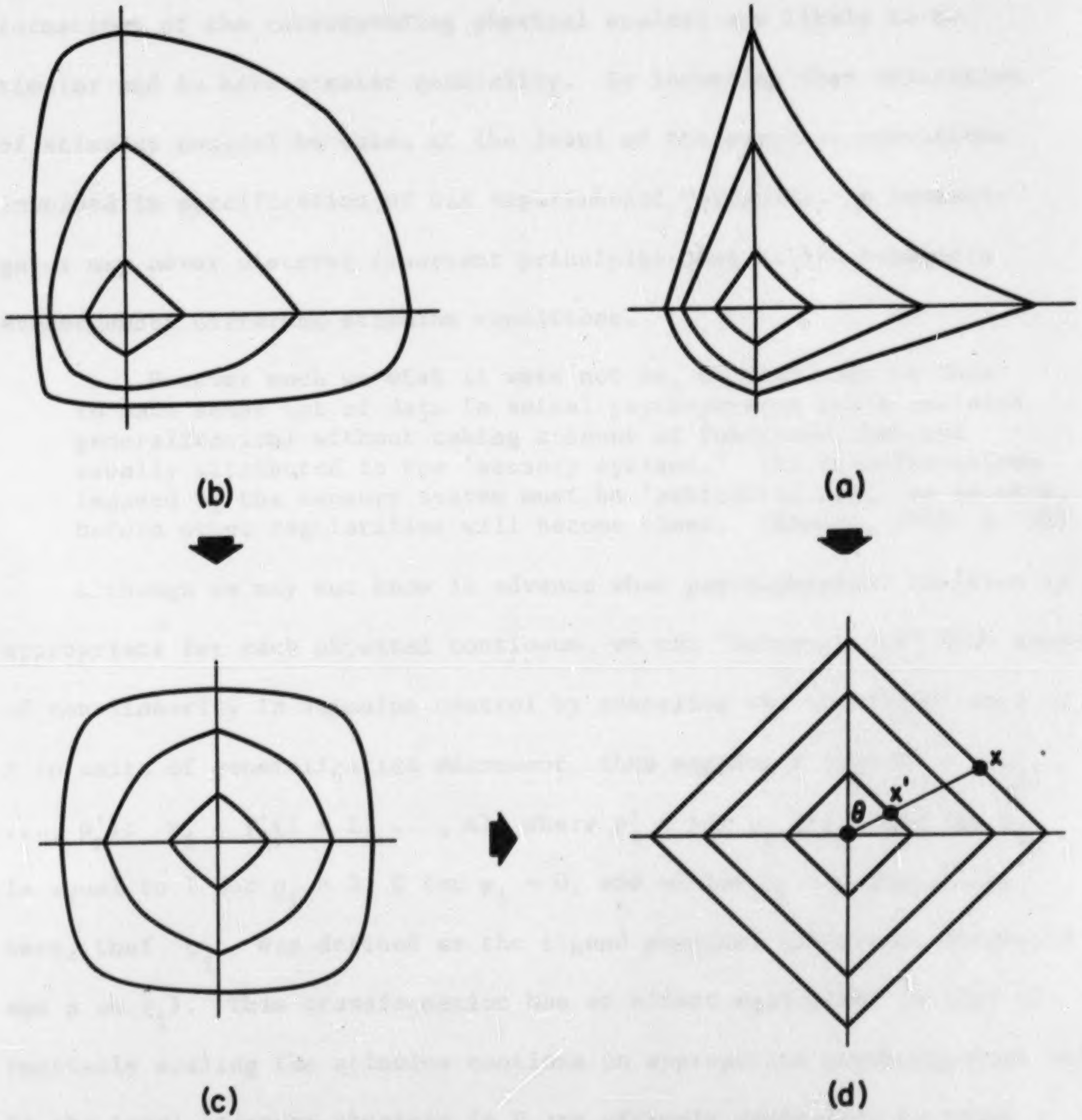

Fig. 1. Level contours arising from nonlinear stimulus-response relations.

of "loudness," "pitch," or "hue" (represented by psychophysical transformations of the corresponding physical scales) are likely to be simpler and to have greater generality. By insisting that principles of stimulus control be taken at the level of the physical operations involved in specification of his experimental variables, an investigator may never discover important principles that relate behaviors evoked under different stimulus conditions.

> However much we wish it were not so, we shall not be able to make sense out of data in animal psychophysics (this includes generalization) without taking account of functions that are usually attributed to the 'sensory systems.' The transformations imposed by the sensory system must be 'subtracted out,' as it were, before other regularities will become clear. (Blough, 1965, p. 35)

Although we may not know in advance what psychophysical function is appropriate for each physical continuum, we can "subtract out" this source of non-linearity in stimulus control by rescaling the coordinate axes of P in units of generalization decrement, thus mapping P into $P' = \{(p'_1, \ldots, p'_n)\colon p'_i \varepsilon P'_i (i = 1, \ldots, n)\}$ where $p'_i = \text{sgn}\, p_i\, \overline{g}(p_i)$ and $\text{sgn}\, p_i$ is equal to 1 for $p_i > 0$, 0 for $p_i = 0$, and -1 for $p_i < 0$ (recalling, here, that p_i was defined as the signed physical difference between θ and p on P_i). This transformation has an effect equivalent to that of initially scaling the stimulus continua in appropriate psychophysical units. If the level contours obtained in P are affinely equivalent to those in Fig. 1(a), then a rescaling of the axes in units of generalization decrement would result in contours in P' affinely equivalent to those in Fig. 1(d). The Minkowski distance from θ to x in Fig. 1(d) is equal to the ratio of the Euclidean lengths $\theta x'$ and θx (where x' lies on the "unit contour").

2. Choice of response scale. Rescaling the stimuli in units of behavioral control forces symmetry on each separate dimension of P' and may impose symmetry in the large, so that $\overline{g}(p') = \overline{g}(-p')$ for all $p' \varepsilon P'$.

However, this outcome does not guarantee that P' will be metrizable in terms of $\overline{g}$. Consider the level contours shown in Fig. 1(b). Rescaling the axes of P in behavioral units transforms P to the space represented in Fig. 1(c) that has level contours which are centrally symmetric and convex, but not homothetic. The shape of these contours changes gradually with level of generalization decrement. This means that the behavioral measures leading to this result cannot serve as Minkowskian distances between stimuli.

In order to succinctly express these hypothetical findings in terms of a suitable metric, it is necessary to transform the original generalization measures, that is, to redefine stimulus control in terms of an alternative response scale. In the example at hand a logarithmic transformation of the dependent variable changes the structure of P as represented in Fig. 1(c) to the simple metric structure depicted in Fig. 1(d). In fact, this is the only transformation that will simultaneously map all level contours of Fig. 1(c) into a set of contours that have, regardless of level, the same centrally symmetric and convex shape. No alternative transformation exists, for example, that will map the contours of Fig. 1(c) into a set of concentric circles. If this outcome is observed with real data the suggested transformation need not be viewed as an arbitrary handling of data, but rather as implying that the response measures are related exponentially (in this particular example) to alternative response measures in terms of which this simple metric structure would have been obtained directly.

Recalling that generalization decrement was defined as $\overline{g}(p) = g(\theta) - g(p)$, where g is the generalization function, we must turn our attention to the measurement operations entering the experimental definition of generalization. It is clear that the form of the empirical generalization gradient and the questions of whether generalization decrement is linear or nonlinear with the size of the stimulus difference in psychophysical units, and of whether a metric combination rule applies, depends on what property of behavior is being measured, and how it is measured. Generalization can be, and has been, empirically defined in terms of a variety of response measures: response probability, number of responses emitted in a fixed interval of time, response latency, rate, amplitude, number of responses to complete extinction, etc. These variables are typically not linearly related, but they are presumed to be mutually interrelated in such a way that if a metric can be established for one, then appropriate transformations on the other measures will result in the same metrization of stimulus control. As Coombs pointed out in connection with the development of mathematical models for analyzing behavioral data:

> The problem is not to be formulated in terms only of transforming these response measures to construct psychological spaces, but rather of transforming them to construct interlocking systems which mutually relate the response measures to each other so that we would obtain the same metric space from each. The models for processing these several measures should be mutually compatible in the sense of yielding at least the same metric relations. (Coombs, 1964b, p. 526-527)

Metric Models for Multidimensional Stimulus Generalization

There are three major points of view concerning rules of combination in multidimensional stimulus generalization. Two-dimensional models of these views are presented graphically in Fig. 2. The two stimulus dimensions are labelled S1 and S2, respectively, and their point of intersection represents the training stimulus. The amount of generalization is depicted as a third dimension, labelled R, orthogonal to the other two. The stimulus dimensions are assumed to be appropriately scaled in units of behavioral control so that the generalization gradients for the separate dimensions are linear and of equal slope. The task of

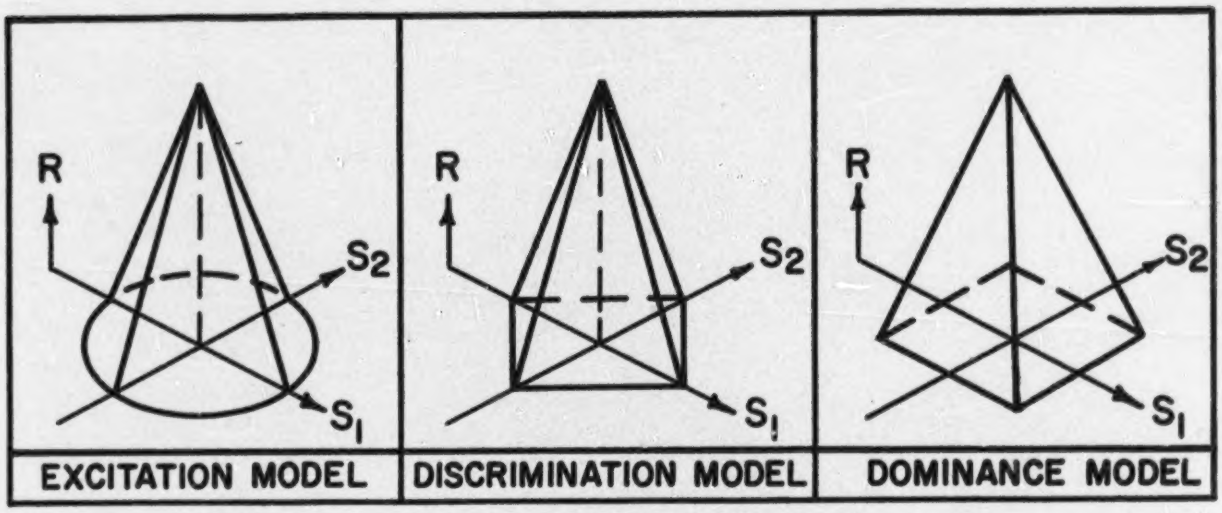

Fig. 2. Possible forms of the generalization surface in two dimensions.

the models is to predict the height of the generalization surface at any point in the four quadrants of the stimulus plane.

Theoretical justification for the first model, the excitation model, comes initially from Pavlov who proposed that generalization was due to a wave of excitation irradiating from the spot on the cortex stimulated as a result of the presentation of a stimulus:

> It may be assumed that each element of the receptor apparatus gains representation in the cortex of the hemispheres through its own proper central neurone, and the peripheral grouping of the receptor organs may be regarded as projecting itself in a definite grouping of nervous elements in the cortex. A nervous impulse reaching the cortex from a definite point from the peripheral receptor does not give rise to an excitation which is limited within the corresponding cortical element, but the excitation irradiates from its point of origin over the cortex, diminishing in intensity the further it spreads from the center of excitation. (Pavlov, 1927, p. 186)

A somewhat over-simplified interpretation of this view is that generalization between two stimuli is a function of the physical distance between their cortical representations. The important point, however, is that generalization is not referred to fixed sensory dimensions but occurs as a "spread of excitation" in all directions in the cortex. Stimuli are presumed to "acquire" the capacity to elicit a response because of certain electro-chemical effects which "spread out" from the initial point of stimulation.

Generalization was given a similar interpretation in Hull's neural-interaction theory. Hull (1943) postulated that generalization represented a "spread of habit strength" from the neural elements involved in stimulation by the conditioned stimulus to those activated by the test stimuli. This "spread of habit strength" from one physical stimulus to another was assumed to occur along innate afferent neural continua as an exponential decay function of the psychophysical similarity of the stimuli, as measured in discrimination units (j.n.d.'s).

These "spread of effect" theories imply that a generalization surface for two dimensions can be generated by rotating a unidimensional gradient around its peak, resulting in a surface which is conical in shape, as depicted in the first panel of Fig. 2. The most important psychological implication of this structure is that for a generalization surface of given dimensionality, one set of orthogonal dimensions, or one frame of reference, is as good as any other in predicting generalization behavior. Each pair of stimuli in the space defines a dimension along which generalization may occur.

According to the views of Pavlov and Hull, prior differential training along specific stimulus dimensions is not necessary to obtain gradients of generalization. A contrasting view, proposed by Lashley and Wade (1946), is that generalization is a performance phenomenon reflecting only an organism's failure to discriminate relevant aspects of a stimulus. They specifically reject Hull's claim that generalization represents a gradient of habit strength developed during conditioning. They argue that if an organism responds (generalizes) to a test stimulus it is only because it has not yet been conditioned to respond differentially to that stimulus difference along the relevant continuum, or it is not attending that aspect of the stimulus. If discrimination occurs, as reflected in generalization decrement, it occurs along stimulus dimensions previously established for the organism by differential training.

This view, according to Guttman (1956), predicts that generalization decrement for any stimulus varying in two dimensions is simply the sum of the decrements occurring along each of the component dimensions. This is the discrimination model depicted in the second panel of Fig. 2 and it is represented by a generalization surface that has the appearance of a pyramid, with its four corners on the stimulus axes. A significant feature of this structure is that it is determined by a fixed set of dimensions which are not subject to arbitrary rotation.

The dominance model is suggested by the results of the experiments, discussed earlier, in which some one stimulus property appeared to be "dominant" in exerting behavioral control. This model states that stimulus control is selective--exerted only by the dimension along which the most discriminable stimulus change occurred. Theoretical background for this model may be found in Lashley's "principle of dominant organization." Lashley (1942) proposed that the mechanism of nervous integration may be such that, when any complex of stimuli arouses nervous activity, that activity is immediately organized and certain stimulus properties become dominant for reaction while others become ineffective. The generalization surface predicted by the dominance model is depicted in the third panel of Fig. 2; it is also a pyramid with its corners at 45° to the stimulus axes.

If we look at successive sections through the generalization surfaces of Fig. 2, sections taken horizontal to the respective stimulus planes, sets of concentric contours are revealed. The contours are circular for the excitation model, diamond-shaped (rotated squares) for the discrimination model, and square for the dominance model. Each contour corresponds to a prescribed amount of generalization decrement and describes a locus of equally effective stimuli. That is, all stimuli on a prescribed contour are equivalent with respect to the amount of control exercised over differential responding, and thus are equally substitutable, one for another, to produce the prescribed generalization decrement. These are the level contours discussed in the preceding section, whose shape uniquely determines the metric appropriate for a given set of measures. Thus, the assumption of the excitation model that all contours are circular is equivalent to the very special assumption that distances, in units of behavioral control, are Euclidean, and the combination rule predicting the generalization decrement to stimuli varying in two dimensions is the square root of the sum of squared decrements occurring on each component dimension. This is the combination rule adopted by most multidimensional scaling models.

The metric underlying the discrimination model provides an engagingly simple rule for combining distances along component dimensions. Namely, the distance between two points is equal to the sum of the differences between the projections of the two points on each of their coordinate axes. This is often called the "city-block" metric because the distance between two points is the total distance that must be traversed in a north-south direction, plus the distance that must be traversed along an east-west direction in order to get from one point to another. Theoretical arguments for this rule of combination have been proposed by Landahl (Householder and Landahl, 1945, p. 76; Landahl, 1945) and by Restle (1959).

The dominance metric requires that the distance between two points is simply the greatest of the distances separating their projections on each of the coordinate axes. Although this combination rule differs markedly from that of the "city-block" metric, in terms of its psychological implications and with regard to the question of how behavioral control is distributed among the component dimensions of a complex stimulus, mathematically the "city-block" space and the "dominance" space are isometric. Their isometry is evidenced by the observation that a 45° rotation of the stimulus plane (stretched by a factor $\sqrt{2}$) maps the level contours of the "city-block" metric into those of the "dominance" metric. Two points are served by this observation: (1) distances are not preserved under rotation of axes, that is, "city-block" distances are mapped into "dominance" distances, thus the dimensions of stimulus control for a given combination rule are unique; and, (2) unless the effective stimulus dimensions are known in advance of analysis, the combination rule generating the data cannot be uniquely determined. Square equal-level contours may imply the appropriateness of a "city-block" combination rule with reference to one set of axes but with reference to an alternative set of axes the "dominance" combination rule is implied.

A General Minkowski Model for Multidimensional Stimulus Generalization. Each of the three models considered for stimulus generalization provides differential predictions of the quantitative properties of behavior under complex stimulus control. It turns out that the metrics associated with these models are special instances of a single parameter family of distance functions known as the Minkowski r-metrics. For any $r \geq 1$ the r-distance between points $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ is defined to be

$$d_r(u, v) = \left[\sum_{i=1}^{n} |u_i - v_i|^r\right]^{1/r}$$

where the index i ranges over the n dimensions of the space and $|u_i - v_i|$ is the distance between the projections of u and v on the i^{th} dimension.

In application to two-dimensional stimulus generalization we equate the measure of generalization decrement from θ to stimulus $p = (p_1, p_2)$ with the distance $d_r(\theta, p)$, hence

$$\overline{g}(p) = d_r(\theta, p) = \left[x^r + y^r\right]^{1/r}$$

where $x = \overline{g}(p_1)$ and $y = \overline{g}(p_2)$. Thus, in this application, x and y are assumed to represent only positive quantities.

In the ordinary (x, y) plane the level contours corresponding to $|x|^r + |y|^r = 1$ are called the "unit circles" (U_r) for different valuations of r. As shown in Fig. 3, U_2 is the Euclidean circle of radius 1; U_1 is an inscribed square with vertices (±1, 0) and (0, ±1). As r increases to infinity, d_r approaches the distance

$$\lim_{r \to \infty} d_r = \max \{|x|, |y|\}$$

Thus, U_∞ is a circumscribed square with vertices (±1, ±1). A r increases from 1 to ∞, U_r deforms in a continuous manner from the square corresponding to d_1 to the square corresponding to d_∞. Intermediate values of r are represented by contours bounded by U_1 and U_∞. The unit circle for any fixed r contains those for smaller r; as shown in Fig. 3, $U_1 \subset U_2 \subset U_4 \subset U_\infty$.

The significance of r as a parameter of multidimensional stimulus control may be understood in connection with the question: How much does each component property of a compound stimulus contribute to the total behavioral control exerted by that stimulus? Or, more precisely: How are each of the effects that are attributed to the separate stimulus components weighted in the combination rule that predicts multidimensional stimulus control? If, for the compound stimulus $p = (p_1, \ldots, p_n)$, which we assume to have only positive components, we let x and x_i represent the positive quantities $\overline{g}(p)$ and $\overline{g}(p_i)$, respectively, then the Minkowski r-metric defined for $\overline{g}$ can be represented as a weighted sum of all component effects, namely

$$x = \Sigma\, \omega_i x_i$$

where $0 \leq \omega_i \leq 1$ $(i = 1, \ldots, n)$, $1 \leq \Sigma\, \omega_i \leq n$, and ω_i is the weight

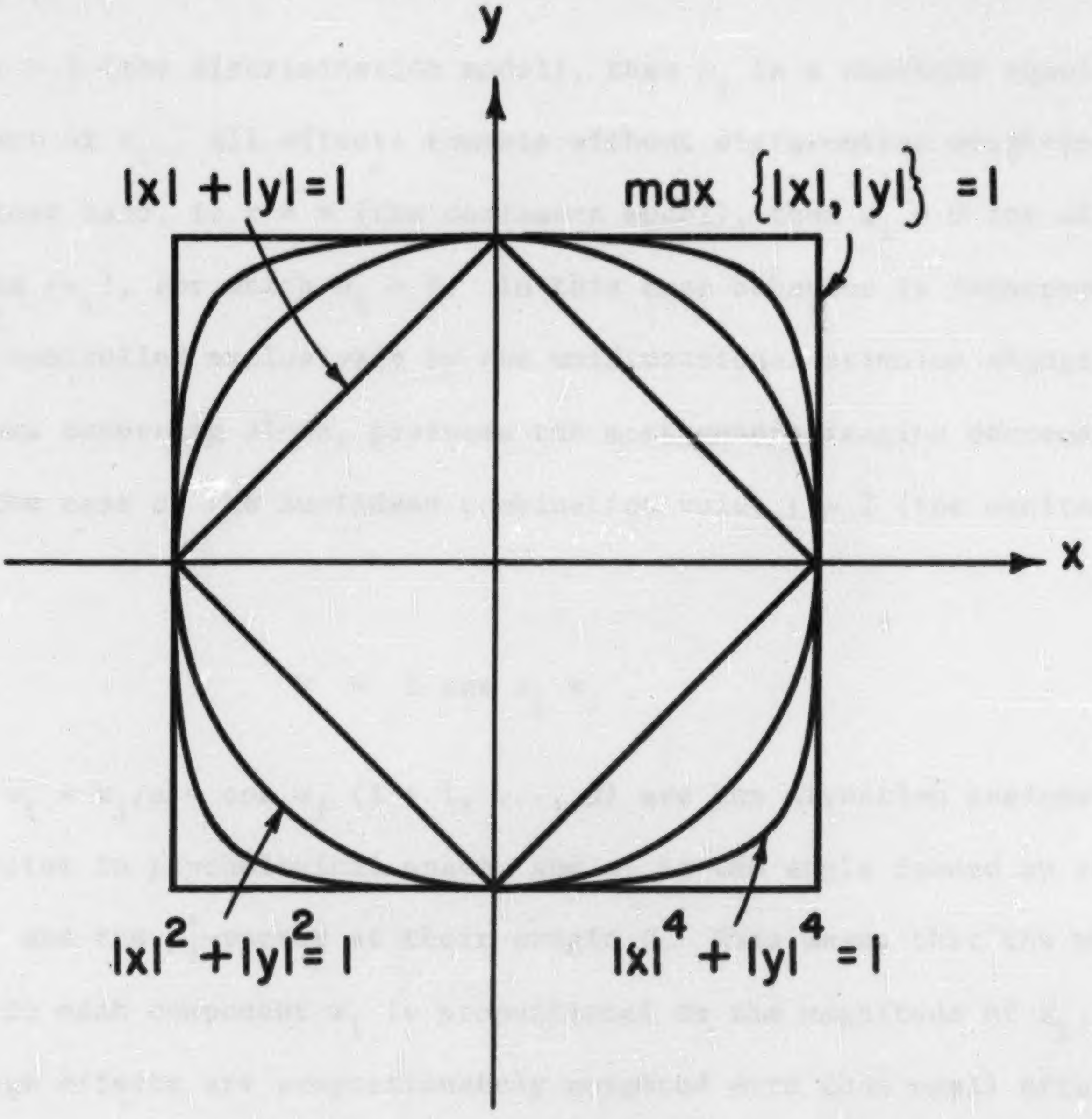

Fig. 3. Euclidean graphs of Minkowski "unit-circles".

associated with x_i and dependent upon x_i in a manner determined by the parameter r.

If r = 1 (the discrimination model), then ω_i is a constant equal to 1 independent of x_i. All effects summate without differential weighting. On the other hand, if $r = \infty$ (the dominance model), then $\omega_i = 0$ for all x_i except $\max_i \{x_i\}$, for which $\omega_i = 1$. In this case behavior is interpreted as being controlled exclusively by the unidimensional stimulus change which, when occurring alone, produces the most generalization decrement.

In the case of the Euclidean combination rule, r = 2 (the excitation model),

$$x = \Sigma \cos \alpha_i x_i$$

that is, $\omega_i = x_i/x = \cos \alpha_i$ $(i = 1, \ldots, n)$ are the direction cosines of the p'-vector in psychological space, and α_i is the angle formed by the p'-vector and the p_i'-vector at their origin θ. This means that the weight assigned to each component x_i is proportional to the magnitude of x_i. Thus, large effects are proportionately weighted more than small effects in the prediction of $\overline{g}(p)$.

For arbitrary r, since $x^r = \Sigma x_i^r$, x can be written as

$$x = \Sigma(x_i/x)^{r-1} x_i \qquad \text{or} \qquad x = \Sigma \, \mathrm{cm}^{r-1} \alpha_i \, x_i \, ,$$

where cm α_i denotes the Minkowski cosine of the angle α_i, pertaining to the relative position of the p_i'-vector to the p'-vector in Minkowski space (see Petty, 1955). (Definitions of trigonometric functions in Minkowski geometry are analogous to those of their Euclidean counterparts. The Minkowski cosine enters some of the same trigonometric identities as does the Euclidean cosine; however it cannot be interpreted as a function of a real number, "the angle formed by arbitrary vectors A and B," because its valuation depends on the order in which the vectors are considered. In general, $cm(A,B) \neq cm(B,A)$; however, if $cm(A,B) = cm(B,A)$ for all A, B in the vector space, then its geometry is Euclidean.)

As r increases, the separate unidimensional effects become increasingly disproportionately weighted by the combination rule that predicts multidimensional stimulus control (to the emphasis of relatively large effects and de-emphasis of relatively small effects). For example, if $0 < x_1 < x_2$, then their respective weights ω_1 and ω_2 are related in the following way:

$$\omega_1/\omega_2 = (cm\ \alpha_1/cm\ \alpha_2)^{r-1} = (x_1/x_2)^{r-1}$$

or

$$\omega_1 = (x_1/x_2)^{r-1}\omega_2 .$$

Thus $\omega_1 < \omega_2$ for any $x_1 < x_2$ and $r > 1$, since $(x_1/x_2)^{r-1}$ is always less than 1 (when $r > 1$) and specifically tends to zero as r increases to $+\infty$. Consequently, as r increases, the weights become increasingly disproportionately distributed over the terms of the summation formula.

This formal result may serve the informal psychological notion that the extent to which an organism "attends" a given unidimensional stimulus change may depend on the changes that occur simultaneously along alternative dimensions of the stimulus. An organism is said to "attend" a particular dimension or property of a stimulus if its behavior is at all under control of variations along that dimension. "Attention is a controlling relation--the relation between a response and a discriminative stimulus.... The criterion [of attending] is whether the stimulus is exerting any effect upon our behavior." (Skinner, 1953, p. 123-124) A given unidimensional stimulus change may be attended when occurring alone but not attended (or attended less) when occurring simultaneously with a more discriminable change along another dimension. In other words, the controlling relations between unidimensional stimulus changes and the response may alter when these changes occur in concert. The dominance metric applies to outcomes in which only a single dimension is attended, namely, that dimension along which the most discriminable change occurred. On the other hand, the city-block metric applies when these controlling relations remain constant, regardless of whether a given unidimensional change occurs alone or in the context of other changes. A Euclidean metrization may be interpreted simply as meaning that the organism attends separately each component dimension of a stimulus in proportion to the relative magnitude of the change in that dimension. By extension, intermediate forms of Minkowski r-metric may represent (by the weights ω_i) other ways in which attention may be distributed among the component properties of a compound stimulus.

Empirical Evidence of the Metric Properties of Multidimensional Stimulus Control

Butter (1963) conducted two operant conditioning experiments with pigeons (10 Ss in the first experiment and 27 Ss in the second), in order to compare generalization to stimuli varied in one and in two dimensions. In both experiments the pigeons were trained to peck at a circular translucent key upon which was projected a narrow (1/16") vertical strip of monochromatic (550 mμ) light. The birds were tested under extinction conditions for generalization to stimuli varied in the wavelength dimension, in the angular orientation (tilt) dimension, and in both dimensions. In the first experiment three levels of wavelength (520, 550, and 580 mμ) were crossed with three levels of angular orientation of the band of light

(40, 90, and 140°), resulting in 9 different stimuli, plus a tenth consisting of a horizontal (0°) band of light (550 mμ). The stimuli were presented separately, 24 times each, for periods lasting 30 seconds. The second experiment was performed in the same manner with nine replications of each of 25 different stimuli, produced by crossing five levels of wavelength (530, 540, 550, 560, and 570 mμ) with five levels of angular orientation (30, 60, 90, 120, and 150°). In both experiments generalization to each wavelength-tilt combination was measured in terms of the average number of responses emitted during the 30-second presentations of the stimulus.

Butter found that relative generalization along the separate dimensions of wavelength and tilt combined multiplicatively to predict relative generalization in both dimensions; that is, the obtained generalization measures entered the following relation:

$$n_{ij}/n_{oo} = (n_{io}/n_{oo}) \times (n_{oj}/n_{oo})$$

where n_{oo} denotes the number of responses emitted to the training stimulus, and n_{io}, n_{oj}, and n_{ij} denote, respectively, the number of responses emitted to a given change in wavelength, angular orientation, and both, respectively. This relation appeared to hold for the responses of individual Ss, as well as for group means in the first experiment. In the second experiment, predictions were less accurate for the results of individual Ss, but the relation held up very well for group means.

The contours that represent equal levels of generalization decrement in a graphic presentation of this combination rule have the shape shown in Fig. 1(c).

Although it was concluded that this finding is not consistent with the view that multidimensional generalization is the result of the algebraic summation of generalization decrements to all discriminable stimulus changes, a logarithmic transformation of the obtained generalization measures renders a simple additive relation between the corresponding measures of generalization decrement (hence, support for the discrimination hypothesis). This means that the response measures serving the definition of generalization in this experiment may enter a simple exponential relation with an alternative response variable, in terms of which multidimensional stimulus generalization has the simple metric properties implied by the discrimination model.

Blough (1965) has pointed out that response rate, that is, "the number of operants emitted in a fixed period of time," may be a poor choice of dependent variable for defining stimulus generalization because it is not a simple measure of response strength but, rather, is a composite measure containing a "number of responses to extinction" component, a "rapidity of response in the presence of the stimulus" component, a "latency-following-stimulus-onset" component, and a "response-probability, having-once-responded" component. How these separate components combine to determine the total number of operants emitted during an interval of time to a generalized stimulus is in no way clearly understood. The practice of averaging response rates over separate presentations of the stimulus (occurring at different times during extinction) and over separate subjects (for whom the response components may interact in different ways) makes the measure even more difficult to interpret simply. In view of these complexities there is no reason to prefer the initial form of this response variable, as a measure of generalization, to a logarithmic function of it--particularly since the latter has the metric properties we desire.

Evidence for a simple additive rule of combination in terms of log response rate was also obtained in a generalization study cited earlier (White, 1958). White trained three groups of 24 kindergarten children to pull a response handle freely during 3.8-sec presentations of a colored stimulus (Munsell patch: 10GY 8/6). Candy was used as reinforcement. One group was tested on the training stimulus and three novel stimuli of the same lightness (value) but having graded differences in hue (5GY, 10Y, and 5Y 8/6). A second group received test stimuli identical to the training stimulus in hue but differing in lightness (10GY 7/6, 6/6, and 5/6). The test stimuli administered to the third group differed from the training stimulus in both hue and lightness (5GY 7/6, 10Y 6/6, and 5Y 5/6). White reported that there was less generalization to stimuli differing in both dimensions than to those differing in either dimension alone. Converting the reported response measures to measures of relative generalization (response rate to training stimulus divided by response rate to test stimulus), it turns out that the generalization functions for the three groups are almost perfectly related multiplicatively, that is, relative generalization to a given change in hue and value equals the product of relative generalization to hue alone, times that to value alone. This relation is the same as that found by Butter.

Numerous studies have been undertaken in the area of multidimensional psychophysics in order to determine what dimensions or psychological attributes of complex stimuli control human judgments of stimulus similarity and dissimilarity, and to derive, from these judgments, scales for the attributes in question. By what principle subjects combine, in a single judgment, differences along several stimulus dimensions has, in general, not been the problem under investigation. Attneave (1950) was the first in this area to investigate directly the combination rule pertaining to multidimensional stimulus similarities. He questioned the appropriateness of the Euclidean assumption underlying multidimensional scaling procedures on the grounds that it implies either that perceptual judgments are not referred to a unique set of psychological dimensions or that in a single judgment subjects combine differences along several stimulus dimensions, according to the square root of the sum of squared differences on each dimension. He viewed both alternatives as objectionable and argued, instead, for an alternative hypothesis which holds that the dimensions of psychological space are unique, and that the perceived difference between two stimuli is equal to the simple arithmetic sum of their differences on the individual dimensions.

Attneave undertook one experiment in which 100 Ss were asked to rate on a seven-point scale the pairwise similarities of seven squares differing in size and/or brightness. The squares were cut from grey paper and mounted in pairs on black cardboard rectangles. Three squares were 1.5 inches on each side with reflectances of 5.3, 8.8, and 37.3 percent, respectively; three squares had a reflectance of 20.7 percent, with sides of 1.0, 2.0, and 2.5 inches, respectively; and a seventh square had sides of 1.5 inches and a reflectance of 20.7 percent. Thus the stimuli formed a "plus" configuration in physical coordinates.

The ratings obtained for the similarity-dissimilarity of each of the 21 stimulus pairs were scaled by the Method of Graded Dichotomies (see Torgerson, 1958). To be considered as "distances", behavioral measures of stimulus differences along any single continuum must be linearly additive. Attneave, however, was unable to fix a zero-point for the judgment scale that would satisfy this condition for the stimulus dimensions involved. Rather, it was found that the judged differences between pairs of stimuli were linear with the differences between the logarithms of their corresponding physical values. Thus the appropriate psychological dimensions appeared to be the log (Fechner) transforms of the corresponding physical continua. In the face of this non-linearity a consistent metric treatment of similarities would have been virtually impossible had not the physical dimensions of the stimuli been known.

Judgments of the overall difference between stimuli differing in both dimensions were found to be related to the physical variables of area and reflectance by an equation of the following form:

$$X_{ij} = W_1 \log(A_i/A_j) + W_2 \log(R_i/R_j) + C$$

in which X_{ij} represents the scaled difference between stimuli i and j; C represents the additive constant, fixing the origin of the judgment scale; W_1 and W_2 are constants, optimally weighting the two dimensions; and A and R represent area and reflectance, respectively. The linear correlation between obtained measures and those predicted by the equation was 0.967.

In log-log coordinates of physical space the level contours corresponding to this outcome are affinely equivalent to the unit circles of the city-block metric.

Similar findings were obtained in two other experiments reported by Attneave: in one, parallelograms, differing in length of base and in angularity, with one also differing in color, served as stimuli; in another, triangles differing in size and in shape served as stimuli. Attneave concluded: "...the psychological differences between [stimuli] may be conceptualized as distances in a non-Euclidean space; this space has an axis system which is psychologically fixed, and hence not subject to rotation in the treatment of data; and a multidimensional distance in this space does not differ greatly from the sum of its projections on the axes." (1958, p. 551)

In the light of this outcome, and in view of the fact that the multidimensional scaling methods (despite their restriction to the Euclidean metric) have apparently achieved satisfactory solutions to data obtained with color stimuli, Torgerson (1958; p. 254) and Shepard (1964; p. 80) suggest that the form of the metric appropriate for a given set of data may depend upon the extent to which the stimuli differ with respect to "obvious and compelling" dimensions, or whether they are perceived as "homogeneous, unitary wholes." Torgerson (1958) pointed out:

> Attneave's choice of dimensions and of the stimuli along these dimensions presented very favorable conditions for obtaining the 'simple sum' type of judgment required by his model. The dimensions varied were simple, obvious, and compelling.... In addition, the stimulus pattern itself [a 'plus' in physical coordinates] would emphasize the 'given dimensions.' (p. 292)

Under these conditions a city-block metric might very well apply. On the other hand, when subjects are confronted with homogeneous patches of color a "different dimension of comparison is likely to be invoked for each pair of colors." (Shepard, 1964; p. 80) Under these conditions a Euclidean metric should be most appropriate.

The difficulty in this analysis lies in the fact that color stimuli do not always result in behavioral data that conform to a Euclidean metric. In White's experiment described above, for example, an additive metric was found to hold for generalization to Munsell color patches. This was also true in the Jones (1962) study. Jones showed a particular Munsell color (7.5PB 5/6) to 30 human _Ss_. Test colors were then presented, varying in saturation (chroma) only, brightness (value) only, hue only, or in all combinations of these properties, taken two or three at a time. Subjects were required to sort the test colors into compartments labelled +, 0, and ?, indicating, respectively, that they were the same as the standard color, different from it, or that _S_ was uncertain whether they were the same or different. A score of 2 was arbitrarily assigned to the test color if it was sorted into the + compartment, a score of 1 if it was sorted into the ? compartment and a score of zero if sorted into the 0 compartment. Three presentations of each stimulus allowed a maximum possible score of 6. A measure of stimulus generalization was defined as the sum of scores assigned to each color averaged over all 30 Ss. Corresponding measures of relative generalization to colors differing from the standard stimulus in only one property were found to combine multiplicatively in predicting relative generalization to colors differing in two or more properties. A metric treatment of these results in terms of log transformations suggests that Attneave's additive model applies to the color domain as well as to stimuli having "obvious and compelling" dimensions.

Summary

A response is said to be under stimulus control when its strength or probability of emission is observed to be greater in the presence of one stimulus than in the presence of another. The amount of control exerted is measured by the change in response strength corresponding to a given change in stimulus value. When the stimuli differ with respect to a single physical property only, the control is ascribed to that property. It is when behavior is under the control of stimuli that vary along two or more dimensions simultaneously that the basic question to which this thesis is addressed arises: namely, what rule of combination predicts the behavioral control exerted by multidimensional stimuli as a function of the control exerted separately by each component property?

The view presented here is that the answer to this question should not be dictated by mathematical convenience, as is the case in conventional approaches to multidimensional scaling, but rather should be viewed as an empirical question allowing that alternative behaviors or behavior evoked under different circumstances or under the control of different kinds of stimuli may conform to different combination rules.

The conditions were discussed under which measures of stimulus control may be treated as Minkowski distances and may be embedded in the particular member of a general class of metric spaces that has the structural properties dictating the combination rule to which the behavior in question conforms so that it may serve both as a multidimensional representation of the stimuli and as a description of the behavior in question. A special class of such spaces is characterized by the single parameter family of distance functions known as the Minkowski r-metrics. These show great promise in application to the problem of metrizing multidimensional stimulus control because, on the one hand, as r takes on the values 1, 2 and ∞ the resulting distance functions represent three contrasting hypotheses concerning multidimensional stimulus generalization. When $r = 1$ the metric implied by the discrimination model is obtained; this model states that the generalization decrement produced by a multidimensional stimulus change is the simple sum of concurrent unidimensional discriminations. When $r = 2$ the excitation model is represented which asserts that generalization results from a "spread" of the effects of reinforcement in all directions about a conditioned stimulus. The dominance model is represented by $r = \infty$; this model specifies that a given unidimensional stimulus change "overshadows" the rest and exerts exclusive control if, alone, it is the most readily discriminated.

On the other hand, each of these three Minkowski metrics--and an indefinite number of intermediate forms--may be regarded as formal hypotheses concerning how each of the effects that are attributed to the separate stimulus components are weighted in predicting multidimensional stimulus control. In this context r is interpreted as a parameter of stimulus control, and as r increases the separate unidimensional effects become increasingly disporportionately weighted by the combination rule that predicts multidimensional stimulus control.

Several quantitative studies of multidimensional stimulus control were reviewed. The collective findings indicate that when the relevant dimensions of control are presumed to be known behavioral measures of multidimensional stimulus control are amenable to metric treatment. This outcome holds for stimulus control established through experimental conditioning by the restriction of reinforcement contingencies to specific stimulus-response relations, as well as for stimulus control established, in the case of human _S_s, through verbal instructions that serve to tap the pre-experimental history of each _S_. Rather than proving the need for alternative metric representations for data collected under different conditions, however, the findings tend to converge on the general appropriateness of a simple additive rule of combination.

CHAPTER II

Attentional Factors in Multidimensional Stimulus Control

Exceptions to the additive combination rule were discussed in an earlier section of Chapter I where it was inferred that the visual discriminations of monkeys conform to a dominance metric rather than an additive metric (Harlow, 1945; Warren, 1953, 1954). In other words, the combination of stimulus properties results not in summation of their separate effects but in an effect equal to that of the property which, alone, is most effective.

This inference is based on the assumption that color was a dominant cue in these experiments because the color differences between the experimental stimuli were more discriminable than the effective size or form differences; it was presumed that if stimuli with sufficiently large size or form differences had been presented the hierarchy of controlling relations would have reversed and size or form would have emerged as dominant cues.

This assumption may be quite invalid and dominant control by color of the visual discriminations of monkeys may occur regardless of the presence of large differences in other stimulus properties. In general, an animal may selectively and exclusively attend stimulus differences along single dimensions regardless of simultaneous variation along other dimensions.

This presents no special problem in the metric treatment of behavioral data if the response surface arising from a multidimensional stimulus space uniformly collapses along irrelevant dimensions of stimulus variation. In the face of this result it is simply concluded that stimulus control is unidimensional. A problem arises when stimulus control is unidimensional but the stimulus property controlling the response changes from one instance to the next or differs between Ss. In pooled data these occurrences are likely not to be detected and the conclusion that stimulus control is multidimensional is misrepresentative of the relations that actually occur in any single S-R episode.

Experimental Evidence for Attentional Factors in Stimulus Control

Cross and Lane (1963) found that when human Ss form identifications for complex tones that differ systematically in both fundamental frequency (f_o) and first formant center frequency (F_1) they selectively attend only one stimulus property (f_o or F_1) depending on the direction in which the stimuli vary in their physical characteristics during training. Twelve Ss were given an identification task requiring the press of a different button in response to each of five different stimuli. For 6 Ss (Group I) the stimuli increased in F_1 directly as they increased in f_o. Thus, during training, Group I Ss were presented stimuli along the principle diagonal of a stimulus matrix and Group II Ss were presented stimuli along the minor diagonal. Otherwise training conditions were identical for the two groups.

The identification structure induced on the extended stimulus set was examined through a generalization testing procedure in which 25 stimuli, comprising all crossings of f_o (110, 130, 150, 170, and 190 cps) with F_1 (514, 819, 1192, 1645, and 2160 cps), were presented 10 times to each S in a randomized schedule.

For each individual S in Group I the relative frequency of emission of each of the five response alternatives was approximately constant across levels of F_1 and varied only with level of f_o. In other words, a correspondence was established between the five response alternatives and the five levels of f_o but F_1 differences were ignored.

The opposite outcome was observed for Group II Ss who disregarded f_o variations and based their identifications only on levels of F_1.

A multivariate uncertainty analysis of the corresponding four-dimensional contingency tables (6Ss × 5 levels of f_o × levels of F_1 × 5 response alternatives) showed, for Group I, that classification according to f_o accounted for 40.0% of the total uncertainty among response alternatives in contrast to only 1.2% uncertainty accounted for by F_1 and 2.4% by classification by S. The information transmitted jointly by all three independent variables accounted for 63.3% of total response uncertainty.

On the other hand, analysis of Group II data showed that f_o classification accounted for only 1.8% of response uncertainty and levels of F_1 accounted for 30.9%. Classification by S accounted for 1.7%. The total information transmitted jointly by all three variables accounted for 53.8% of the total uncertainty in response classification.

Selective attention to single aspects of a stimulus situation has also been observed in the discriminative behavior of animals. Jenkins & Harrison (1960) trained pigeons to keypeck with reinforcement contingent on the presence of a 1000 cps tone and the chamber lights on (S+). The absence of tone and a darkened chamber together constituted S-. Subsequent generalization testing with chamber lights on evoked equal levels of responding (flat generalization gradients) across several values of tonal frequency and also under the condition in which no tone was present. Thus, the auditory component of the stimulus situation exerted no control over responding; the visual component exerted exclusive control.

Newman (in a study described by Baron (1965) also found that a single component of a compound stimulus gains control of a pigeon's keypecking behavior in a successive discrimination. Four groups of birds were trained to keypeck in the presence of a white vertical line on a green background

(S+). For one group S- was the absence of S+. For a second group S- was a green background with no line (the discrimination had to be learned on the basis of presence or absence of the vertical line). For a third group S- was a red background with a line (the discrimination had to be learned on the basis of color). For the fourth group S- was a red background with no line (the discrimination could be learned on the basis of either color or the presence or absence of the line).

The discrimination was established for all groups of Ss. They were then tested for generalization in the presence of lines varied in orientation on a neutral grey background. Evidence that the position of the line exerted differential control over responding was found only for the group that had to learn the discrimination on the basis of presence or absence of the vertical line. Level of responding was suppressed and equal at all orientations of the line for the other three groups indicating that the initial discrimination was based only on color.

These experimental findings clearly establish the fact that selective attention to component properties of compound stimuli may occur and that not all aspects of the stimulus situation present when a reinforced response occurs subsequently provide the occasion for the emission of that response.

There is also evidence that the controlling relations between stimuli and responding may differ from one instance to the next or for different Ss. It has been found in several animal experiments that different aspects of a stimulus may be important in controlling the behavior of different Ss, or of a single S during separate presentations of the same stimulus. For example, Revesz (1925) trained monkeys in a 4-box Yerkes Multiple-Choice apparatus on a simultaneous discrimination in which responses to a yellow circle (S+) were reinforced and responses to a blue rectangle, a red triangle, or a green trapezium, presented simultaneously with S+, were not reinforced. When the same four colors were presented on other forms no S responded to a stimulus that was not identical to S+ in either color or form. On some trials, however, it was found that the response was under control of the form of the stimulus and on others it was under control of its color. Attention appeared to alternate between the two stimulus properties.

Lashley (1938, 1942) showed that when rats form a simultaneous discrimination different aspects of the discriminative stimulus gain control over responding for different rats. Reynolds (1961) obtained the same results. He trained two pigeons on a successive discrimination in which S+ was a white triangle on a red background and S- was a white circle on a green background. When the stimulus components were presented separately (that is, white triangle alone, red background alone, etc.) one S responded only when the white triangle was presented, and the other S responded only when the red background was presented. Thus, only one aspect of S+ acquired control of responding, but it was a different one for each S.

Shepard (1964) investigated the "match to sample" behavior of human Ss in an attempt to determine the shape of the isosimilarity contour and, by implication, the particular Minkowski metric that determines how differences in size and tilt combine to control choices based on overall similarity. Schematic one-spoke wheels (circles with single radial lines) served as stimuli. The dimensions of variation were the diameter of the circle and the inclination of the radial spoke.

The standard stimulus was 3/4" in diameter with its spoke inclined 45° from horizontal. Twenty-four series of 15 different comparison stimuli were presented to each of 60 Ss with instructions to pick the one stimulus from each series that had the "closest overall resemblance" to the standard stimulus. In physically-defined space with coordinates of size and inclination the 15 stimuli in each series were uniformly spaced along straight lines tangential, at 24 different clockwise positions, to an imaginary circle with a center corresponding to the standard stimulus. The middle stimuli in the 24 series were all equidistant (in a physical sense) from the standard stimulus.

The distributions of frequency of choice were plotted separately for each of the 24 stimulus arrays. These distributions appeared to pass through four cycles of change as the test arrays rotated clockwise around the standard stimulus. At the four positions corresponding to unidimensional stimulus variation (constant size, varying tilt; or constant tilt; varying size) the distributions were sharply peaked at the middle stimulus of each array. As the arrays were rotated through positions at 45° angles to the physical axes (corresponding to simultaneous variation in both size and tilt) these distributions flattened out considerably; in fact, they appeared to become slightly bimodal. This result implied that the isosimilarity contour had four regions of sharpest curvature. Instead of conforming to a Euclidean ellipse, it presumably resembled a roughly "four-cornered" figure. In addition, the tendency toward bimodality of the distributions of choices along the four tangential series that ran at 45° through the physical representation implied that the "four-cornered" contour must in fact have been concave, indicating a breakdown of metric structure.

Shepard suggested that this anomalous result was due to the indiscriminate pooling of data from subjects who were attending different aspects of the stimuli. The Ss were divided into two groups; the data from Ss whose behavior appeared to be under the control of both stimulus properties were analyzed separately from the data of those whose behavior appeared to be predominantly under the control of one or the other stimulus property. The Ss in the latter group were described as adopting a "matching strategy" which essentially involved "finding the best match along one of the two dimensions." Data from Ss of the other group, who were presumed to be attending both dimensions, indicated that the isosimilarity contour was convex, but still roughly "four-cornered," indicating the appropriateness of a Minkowski metric somewhere between the Euclidean and the city-block varieties.

In the second experiment, similarity was defined in terms of the proportion of times that two stimuli were confused during identification training. As in the preceding experiment, each stimulus consisted of a circle of a certain size, together with a single radial line inclined at a certain angle. Only eight stimuli were used, however, and they were chosen so as to form the vertices of a regular octagon in the two-dimensional physical representation of the stimuli. Twenty Ss were required to learn a different, arbitrarily specified letter of the alphabet as a label for each of the eight stimuli. The errors made during the learning of these stimulus-response assignments provided a "measure of similarity" for each pair of stimuli, based on the number of times either stimulus in the pair led to a response assigned to the other stimulus in that pair.

Shepard reasoned that "the assumption of a Euclidean metric would predict that about the same number of confusions should occur between any two stimuli that are represented by adjacent points around the octagon" because the two dimensions were scaled with respect to each other in such a way that equal displacements along either physical dimensions would produce equivalent psychological changes. Even if some eccentricity remained in the isosimilarity contour the "number of errors between adjacent stimuli should pass through only two cycles as one revolution is completed about the octagon." On the other hand, the assumption of a city-block metric would predict that the number of confusions between the four pairs of adjacent stimuli that differed along only one dimension would always exceed the number of confusions between the four pairs of adjacent stimuli that differed along both dimensions at once. In this case the number of errors between adjacent stimulus pairs should pass through four cycles as one revolution is completed about the octagon.

The outcome was strongly in support of the second of these two predictions. Shepard found, however, that this outcome (based on averaged data) was not representative of the results that appeared to hold for individual S̲s. A correlational analysis showed that S̲s who frequently confused stimuli differing along one dimension on one side of the octagon also tended to confuse stimuli differing along that same dimension on the opposite side of the octagon, but confusions between stimuli differing along the other dimension were not correlated across sides. Approximately half of the S̲s tended to confuse differences in size predominantly, and the other half confused differences in tilt. Thus, the metric suggested by the averaged data resulted from pooling the responses of S̲s who were attending different properties of the stimulus.

These results lead us to suspect that attentional factors may have been operating in other experiments only to be disguised by the practice of averaging data. If attentional factors are operating, averaged data is likely to have characteristics not to be found in individual data and it is likely not to be representative of momentarily effective S-R relations. This does not mean that valid inferences cannot be drawn from results based on averaging, but it does mean that the nature of these inferences and the interpretation of results must be consistent with the facts of attention. In other words, in interpreting results it must be allowed that measures may not represent the homogeneous effects of all stimulus properties operating simultaneously but rather the composite effect of pooling separate S-R occurrences.

The occurrence of different controlling relations between the stimulus and response upon different presentations of the same physical stimulus during generalization testing does not preclude the possibility of achieving a multidimensional metric solution for averaged data. Indeed, conditions are discussed below under which a simple additive metric for multidimensional stimulus generalization defined in terms of response probability follows from the explicit assumption that the component properties of stimuli are selectively attended and that for any single S-R episode the response is controlled by only one discriminative aspect of the stimulus.

Attention and the Metric Structure of Pooled Data

Generalization testing procedures have been used in both experimental conditioning and psychophysics to obtain information regarding, on the one hand, what properties of a stimulus are important in controlling behavior and, on the other hand, how much control is exerted by a stimulus that differs (with regard to these properties) from the stimulus initially designated as the S^D or as the standard. Because of the unreliability of single observations, the measures of stimulus control obtained in these studies are invariably based on pooled data, whether the pooling is done over Ss or over separate presentations of the stimulus for a single S, or both. If we allow that different controlling relations may occur between the separate components of the stimulus and the response upon different presentations of the stimulus, or for different Ss, (as is the case when a subject's attention shifts from one stimulus property to another upon different stimulus presentations or when the data are indiscriminantly pooled over Ss who are attending different properties of the stimulus) then the resulting measure of stimulus control is actually a composite measure which is, in principle, analyzable into separate components each of which represents a separate and distinguishable outcome with respect to the controlling relations that might obtain between stimuli and a response.

We will consider these outcomes in the context of a stimulus generalization study wherein generalization is operationally defined and measured in terms of the relative frequency of stimulus presentations that lead to R, a response appropriate to the training stimulus.

We assume that generalization represents a failure of discrimination; that is to say, if a test stimulus, S, differs from the training stimulus, θ, with respect to, say, two properties X and Y, then the evocation of R by S implies that neither the difference in X between S and θ nor the difference in Y exercises differential control over behavior. On the other hand, if R is not evoked by S (or an alternative response not conditioned to S is evoked) then a discrimination is said to occur and it is assumed that this can happen in either of three ways: differential control may be exerted by the difference in X alone, in Y alone, or in both X and Y simultaneously. In accordance with the previous definition of "attention" these outcomes may be equated with the following statements: (1) S fails to attend the difference in either X or Y, (2) S attends the difference in X but not in Y, (3) S attends the difference in Y but not in X, (4) S attends the difference in both properties X and Y simultaneously. The first outcome results in generalization and the remaining three result in discrimination.

These comprise four mutually exclusive and exhaustive events with regard to the controlling relations that can obtain between a response and two-dimensional stimuli. Accordingly, the set of N experimental presentations of a given stimulus can be partitioned into four distinct subsets so that

$$N = n_1(XY) + n_2(X\bar{Y}) + n_3(\bar{X}Y) + n_4(\overline{XY}) \tag{1}$$

where

$n_1(XY)$ = the number of S presentations resulting in generalization.

$n_2(X\bar{Y})$ = the number of S presentations resulting in discrimination under control only of Y.

$n_3(\bar{X}Y)$ = the number of S presentations resulting in discrimination under control only of X.

$n_4(\overline{XY})$ = the number of S presentations resulting in discrimination under control of both X and Y.

Taking the relative frequency of each of these joint events as a measure of the probability of its occurrence (1) becomes

$$1.00 = P(XY) + P(X\bar{Y}) + P(\bar{X}Y) + P(\overline{XY}). \tag{2}$$

We assume that R is consistently evoked by θ so that generalization decrement to S can be defined as 1 - P(XY).

If it is not allowed that the separate properties of the stimulus may be selectively effective in controlling behavior and that multidimensional stimulus control always represents the combined effect of all stimulus differences then $P(X\bar{Y}) = P(\bar{X}Y) = 0$ and generalization decrement to S is equal to the probability of the joint event of both properties exerting differential control, that is

$$(3) \qquad 1 - P(XY) = P(\overline{XY}).$$

On the other hand, if it is allowed that the separate properties are selectively attended and that discrimination is never under the control of both properties simultaneously then

$$(4) \qquad 1 - P(XY) = P(\overline{X}Y) + P(X\overline{Y}).$$

Nonzero measures for both of the two terms on the right imply that either a subject's attention shifts from one stimulus property to another upon different presentations of the stimulus or the data are indiscriminantly pooled over *S*s who are attending different properties of the stimulus.

Assuming the absence of interaction effects, namely, that for a given ΔX, $P(\overline{X}Y) = P(\overline{X}) = [1 - P(X)]$ is constant regardless of the size of ΔY then (4) can be written in general form for all S_{ij} as

$$(5) \qquad 1 - P_{ij}(XY) = [1 - P_i(X)] + [1 - P_j(Y)]$$

where i indexes the difference between S_{ij} and θ with respect to X and j indexes the difference between S_{ij} and θ with respect to Y.

This outcome means that the generalization decrement produced by a stimulus changed with respect to two properties equals the sum of the decrements produced by stimuli differing from θ with respect to a single property only--the prediction made by the discrimination hypothesis and represented by the city-block metric. In this case, however, a two-dimensional metric structure does not represent the homogeneous effect of both stimulus properties exerting simultaneous control but, rather, it arises from the pooling of linearly independent unidimensional metric relations. A discrimination is evoked because either X or Y is exclusively attended and the particular property exerting control depends not on the relative discriminability of the two properties but is, instead, a function of extraneous factors. For example, a subject's prior reinforcement history with regard to the dimensions involved may result in conditioned responsivity to stimulus variations along only a selected dimension. Fluctuating sensory processes such as those underlying the perception of reversible visual figures, like the Necker cube, might produce shifts in attention from one stimulus property to another on separate occasions.

If, on the other hand, the controlling relations are such that the property attended depends strictly on the relative discriminability of the component properties in such a way that $P_{ij}(X\overline{Y}) = 0$ if $P_i(\overline{X}) > P_j(\overline{Y})$ then

$$1 - P_{ij}(XY) = \max \{P_i(\overline{X}), P_j(\overline{Y})\} \tag{6}$$

which is the prediction made by the dominance model. It is recalled that equations (5) and (6) respectively represent the two limiting cases of the Minkowski r-metric discussed in Chapter I, and that these have related structural properties--that is, both have square unit-level contours that differ only in their orientation and extension in the coordinate space.

Interaction of Continua and Effective Dimensions of Control. The derivation of an additive metric for behavioral data arising from the indiscriminate pooling of different unidimensional S-R occurrences involves the assumption that the effective dimensions of stimulus control are collinear with the physical dimensions of the stimulus space. In other words, the level contours pertaining to a given state of attention are lines parallel to one of the axes of the stimulus space. This condition is also required by the dominance metric.

Departures from this condition may occur as a result of an interaction between the physical properties of the stimuli that gives rise to effective dimensions of control that are not related in a simple way to the physical parameters of the space. The Fletcher-Munson equal-loudness contours in tonal space is a well known example of this. The apparent loudness of a pure tone is not a simple function of its physical intensity but depends as well on tonal frequency. If Ss are attending the loudness of pure tones in a discrimination task, then a given change in intensity at one level of tone frequency represents a loudness change different from that corresponding to the same intensity change at a different frequency level. Similarly, pitch is an attribute of sensory behavior that corresponds closely to tonal frequency but is also dependent upon intensity level so the equal-pitch contours are not lines parallel to the intensity coordinates in physical space.

Either of the "selective-attention" models discussed above may give rise to an empirical generalization surface on the tonal space with structural properties characteristic of neither the discrimination model nor the dominance model. If pitch and loudness are the effective dimensions of behavioral control then the isosimilarity contours about a given standard stimulus may be neither diamond nor square shaped but will be determined by the shape of the corresponding pitch and loudness contours. Thus, an entirely different Minkowski metric may be implied; or the presumptive interstimulus distances may exhibit violations of the homogeneity or triangle inequality properties of true distance.

To complicate the problem even further, the number of potentially effective dimensions of stimulus control may exceed the number of physical variables that define the stimulus space. For example, pure tones differing only in frequency and intensity may be compared not only on the basis of loudness or pitch but also on the basis of volume, brightness, density or chroma (referring to the octave effect). Each of these attributes represents a different ordering of the elements of the tonal space;

the level contours corresponding to each attribute have different shapes and directions when plotted in coordinates of frequency and intensity.

In addition, the contours corresponding to levels of a given attribute for one listener may differ appreciably from contours characteristic of a second listener (Ross, 1964). Thus, the discriminative responses of two listeners, presumably attending the same tonal attribute, may imply different dimensionalizations of the tonal space with respect to the direction in which control is exerted.

The shape (in a two-dimensional coordinate space) of the isosimilarity contours about a given reference stimulus will depend upon the number of different directions in the space in which control is exerted. The four-cornered contours characterizing the additive and dominance models and representing two directions of control deform into six-cornered contours with the introduction of a third direction of control, or into an eight-cornered contour with the introduction of a fourth. In general, if there are n possible reference directions in a two-dimensional stimulus space the controlling relations characterizing the two selective-attention models for pooled data will combine to determine polygonal isosimilarity contours with 2n vertices or regions of sharpest curvature. As n increases to the limiting case wherein every potential stimulus pair represents a different dimension of comparison the contours will converge to Euclidean circles.

Thus, a breakdown of simple metric structure for pooled data collected under conditions of selective attention to single stimulus properties may simply indicate the lack of dominant directions of control--that is, the emergence of alternative dimensions of comparison that may be peculiar to individual Ss and that differ in direction in the physical space. Under these circumstances the Euclidean metric may serve as a better approximation to the data than either of the metrics represented by equations (5) and (6).

Summary

There is an accumulation of evidence that the component properties of compound stimuli are selectively effective in the control of discriminative behavior and that control may be exerted by different stimulus properties for different Ss or for the same S upon different presentations of a stimulus. Thus, measures of stimulus control based on pooled data--whether the pooling is done across Ss, stimulus presentations, or both--are likely not to represent the effect of all properties simultaneously exerting control but, rather, the composite effects of pooling mutually exclusive unidimensional S-R relations. This fact does not preclude a metric representation of the data. In point of fact, it was shown that when the effective dimensions of control are collinear with the physical dimensions of the stimuli and that when a discrimination is under exclusive control of one or another dimension upon any one stimulus presentation and never under simultaneous control of two or more dimensions then discrimination probabilities conform either to an additive or a dominance combination rule depending on whether the dimension attended is a function of the relative discriminability of the alternatives.

A breakdown of these simple structures may occur through an interaction of stimulus continua or the availability of alternative dimensions of control that are not collinear with the physical dimensions of the stimuli. Under these conditions a Euclidean metric may serve a close approximation to the empirical structure of the data.

CHAPTER III

An Experimental Comparison of Alternative Measures of Stimulus Control Under Conditions of Selective Attention To Single Stimulus Properties

Introduction

Stimulus control can be, and has been, empirically defined in terms of a variety of response measures: response probability, latency, rate, etc. These measures are typically not linearly related but they are presumed to be mutually interrelated in such a way that if a metric can be established for one, then appropriate transformations of the other measures will result in the same metric combination rule. Metric treatment of the data from three generalization studies reviewed in Chapter I required, in each case, a logarithmic transformation of the experimental response measures (average rate of emission of a free operant in two cases and what was essentially a 3-point category scale measure in the third case). It was argued that the multiplicative structure of the data implied that these measures were related exponentially to alternative response measures in terms of which a simple additive metric would have been obtained directly. In the absence of research designed to simultaneously compare alternative measures of stimulus control this point is only argumentative and there is no reason (short of the theoretical ones) to prefer the metric combination rule over the multiplicative rule as a general principle of behavior.

The purpose of the present study is twofold: (1) the comparison of response probability and average response latency as measures of two-dimensional stimulus control and (2) comparison of the metric structure of pooled data under conditions in which Ss are free to attend both stimulus dimensions with the metric structure arising from conditions in which selective attention to one or the other dimension is under strict experimental control.

The technique employed to establish experimental control of the property attended is a discrimination training procedure in which the pattern of reinforcement for the emission of either of two mutually exclusive responses to successive presentations of three two-dimensional auditory stimuli (S, $S_{\Delta x}$, $S_{\Delta y}$) is contingent upon the presence of one or two visual cues (A or B) in accordance with the following stimulus-response assignments:

$$(S|A) \rightarrow R_1, \qquad (S_{\Delta x}|A) \rightarrow R_1, \qquad (S_{\Delta y}|A) \rightarrow R_2$$

$$(S|B) \rightarrow R_1, \qquad (S_{\Delta x}|B) \rightarrow R_2, \qquad (S_{\Delta y}|B) \rightarrow R_1$$

Under this paradigm Ss are trained to generalize one dimension of stimulus change and discriminate the other in the presence of one visual cue and to reverse this relation in the presence of the alternate cue.

The structure induced on the extended stimulus set can be examined, using generalization testing procedures with the two visual cues presented singly (to control selective attention) and in combination (setting the occasion for simultaneous control by both dimensions).

METHOD

Subjects

Twenty male University students served as Ss in individual sessions lasting approximately 80 minutes. None had previous experience as a participant in a generalization or discrimination experiment.

Stimuli and apparatus

The stimuli were frequency modulated sinusoidal tones varying in both center frequency and the rate at which they were frequency-modulated ±20 cps. The tones were generated by a beat frequency oscillator with a built-in saw-tooth modulator (Bruel and Kjaer 1014) and pre-recorded on magnetic tape (Ampex 351-2U) at four different levels of center frequency (cf), 500, 600, 750, and 900 cps, and of modulation rate (mr), 2, 4, 8, and 16 sweeps/sec (sps). All combinations of center frequency with modulation rate were used, yielding a set of 16 different stimuli.

The experimental space was a sound-attenuating room containing a rack-mounted panel and shelf. On the front of the panel were two pilot lamps of different colors, amber and blue, and a tray for receiving pennies. An electrically-operated coin dispenser, obtained from a vending machine and modified for experimental use, was mounted behind the panel. A response key (Poucel Electronics) that could be pressed either to the left or to the right was mounted on the shelf. The key was adjusted so that a lateral displacement of 1/16" in either direction (left or right) would actuate the response mechanism. The chamber was dark except when one or both of the pilot lamps was lit. The control equipment was in an adjoining room.

The stimuli were reproduced by the tape recorder and presented to S through calibrated earphones (Grason-Stadler TDH-39). Interposed between the tape recorder and the earphones was an electronic switch (Grason-Stadler 829D) that closed upon presentation of the stimulus and opened when the response key was pressed thus terminating the stimulus. The electronic switch, as well as occasions for reinforcement and other experimental events, was controlled by voice-operated relays (Miratel), which were operated, in turn, by coding tones synchronized with the stimuli and pre-recorded on a second track of the magnetic tape. Response latencies were measured by recording, on a second tape recorder (Ampex 601), 1000 cps tones passed through a second channel of the electronic switch. The duration of these tones corresponded exactly to the time interval from the onset of the stimulus to its termination by a response. On subsequent play-back these durations were measured by a frequency counter (Hewlett-Packard) and automatically punched on IBM cards for analysis.

Procedure

The Ss were seated individually in the experimental chamber and the following instructions were read:

> You can earn money in this experiment if you can learn to correctly respond to certain properties of sounds that you will hear in these earphones. You will respond by pressing this key either to the left or to the right. If your response is correct, a penny will be dispensed to you; if it is not, you will get nothing. In either case, the response you make, right or wrong, will terminate the stimulus. The basis for a correct discrimination will depend upon which of these two lights is on at the time. The faster you learn and the fewer errors you make, the more money you will earn.

Discrimination training. Only three different stimuli were presented to S during the training phase. The actual combinations of center frequency and modulation rate used were the following: (500 cps/2 sps), (500 cps/16 sps) and (900 cps/2 sps). In the following schematic representation of the complete stimulus set the training stimuli are identified by the ×'s.

mr (sps): 16 →	×	.	.	.
8 →	.	.	.	.
4 →	.	.	.	.
2 →	×	.	.	×
	↑	↑	↑	↑
cf (cps):	500	600	750	900

These three stimuli were presented to S in 10 randomized sequences each comprising 15 stimulus presentations (five presentations of each training stimulus) for a total of 150 stimulus presentations. During five sequences the blue light was on. (The order was randomized.) Under either light condition, a 500 cps/2 sps signal was an S^D for the response of pressing the key to the left, R_1. A 500 cps/16 sps signal was also an S^D for R_1 when the amber light was on, but it was an S^D for a key press to the right, R_2, when the blue light was on. A 900 cps/2 sps signal was an S^D for R_2 when the amber light was on, and for R_1 when the blue light was on. These stimulus-response assignments for the two light conditions are summarized below.

	AMBER LIGHT ON	BLUE LIGHT ON
500cps/ 2sps →	R_1	R_1
500cps/16sps →	R_1	R_2
900cps/ 2sps →	R_2	R_1

The stimuli were presented at 10-sec intervals for a maximum duration of 5 sec on each presentation. If at the end of 5 sec S did not respond, the stimulus terminated and the reinforcement mechanism was inoperative. Otherwise the stimulus was terminated by the response, whether it was correct or not. If correct, a penny was dispensed and a bell, with a sound similar to that of a cash register, was rung. There was a 30-sec time-out period between each sequence of 15 stimulus presentations. During these periods Ss were in total darkness.

Generalization testing. At the completion of training, the experimenter re-entered the chamber and told S that, for the remainder of the experiment, the coin dispenser and bell would be disconnected "in order to test whether [S] could continue to respond appropriately without feedback as to the correctness of [his] response." In addition, S was instructed to respond to each stimulus. All Ss were presented 10 randomly permuted sequences of the 16 test stimuli with a 30-sec time out period between each sequence. Within each sequence the stimuli occurred at 10-sec intervals for a maximum duration of 5 sec on each presentation if S failed to respond. Responses with latencies shorter than 5 sec terminated the stimulus. For ten Ss (Group I) the amber light was on during five of the presentation sequences and the blue light was on during the other five in a randomized order. For the other ten Ss (Group II) two of the first four stimulus sequences were presented with the amber light on alone and the other two with the blue light on alone. For the remaining six stimulus sequences, however, both the amber and the blue light were on. Thus, Group II Ss served as their own controls with regard to the comparison of the results of generalization testing under conditions of selective attention to single stimulus properties with the results of testing under conditions setting the occasion for simultaneous control by both stimulus properties.

Results

Discrimination training. The course of development of stimulus control was rapid for all 20 Ss. The median number of incorrect responses evoked in the course of 150 stimulus presentations to each S was only 6. As few as 2 errors were made by one S and the greatest number of errors made by a single S was 17. Nearly all errors were made during the first four stimulus sequences. This means that the stimulus property attended by each S was readily brought under experimental control of the visual variable. Ss learned to emit R_1 consistently in response to the 500 cps/2 sps tone under both light conditions and to respond differently (emit R_2) only to the center frequency (cf) difference when the amber light was on and only to the modulation rate (mr) difference when the blue light was on.

Generalization testing: Group I. The transfer of stimulus control to stimuli varying simultaneously in both cf and mr was measured in terms of R_1 (generalization) and R_2 (discrimination) frequency and also in terms of R_1 and R_2 latency. Generalization gradients of R_1 frequency for Group I are shown in Fig. 4 as a function of cf, with mr as the parameter, and as a function of mr, with cf as the parameter, under both the amber and the blue light conditions. Each data point represents the total number of R_1 responses evoked in the course of 5 presentations of each test stimulus to each of the 10 Ss in Group I for a maximum frequency (pooling over stimulus presentations and over Ss) of 50. The top graphs in Fig. 4 show clearly that when the amber light was on differential responding was exclusively under the control of changes in cf; changes in mr did not produce significant changes in R_1 frequency. On the other hand, the bottom graphs show that mr exercised exclusive control when the blue light was on and that changes in cf were not attended.

The dependence of auditory stimulus control on the color of the visual cue is represented by light condition × center frequency and by light condition × modulation rate interaction effects in an analysis of variance of total R_1 frequency. These effects prove to be extremely significant (see Table 1). On the other hand, the variance attributed to interaction of the primary stimulus dimensions (cf × mr) is not significantly different from error variance ($F = 1.26$), indicating that the differential main effects of cf and mr variation combine additively across light conditions in determining the total differential effect of simultaneous change in both stimulus dimensions. This total effect is measured in terms of total R_1 probability, as estimated by the relative frequency of R_1 to all presentations of a given stimulus (pooling over Ss and over light conditions).

It is evident from these results that experimental control of attention was achieved and that, under conditions of selective and exclusive

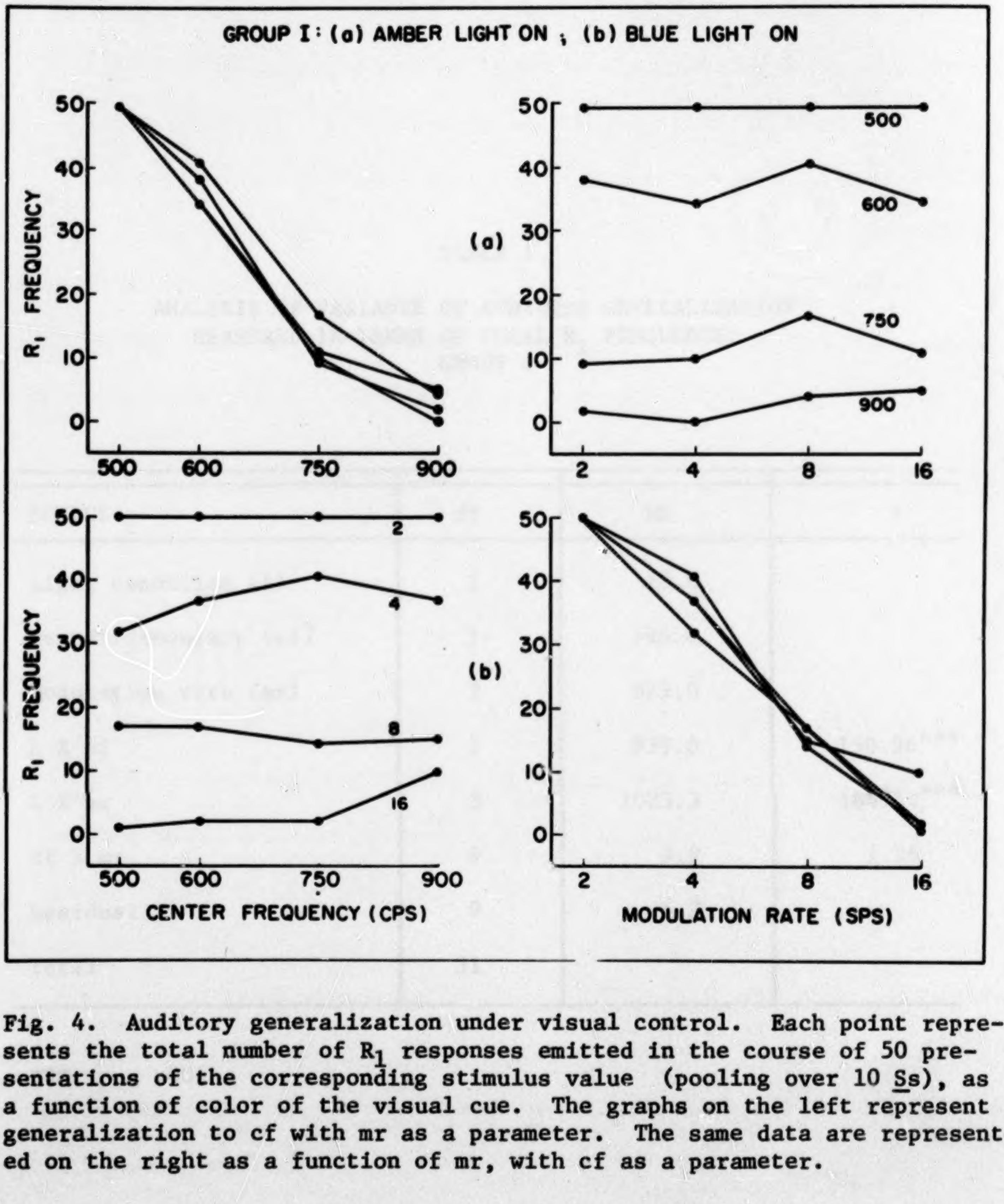

Fig. 4. Auditory generalization under visual control. Each point represents the total number of R_1 responses emitted in the course of 50 presentations of the corresponding stimulus value (pooling over 10 Ss), as a function of color of the visual cue. The graphs on the left represent generalization to cf with mr as a parameter. The same data are represented on the right as a function of mr, with cf as a parameter.

TABLE 1

ANALYSIS OF VARIANCE OF AUDITORY GENERALIZATION MEASURED IN TERMS OF TOTAL R_1 FREQUENCY: GROUP I

SOURCE	df	MS	F
Light condition (L)	1	18.0	
Center frequency (cf)	3	796.0	
Modulation rate (mr)	3	829.0	
L X cf	3	939.0	150.96***
L X mr	3	1023.3	164.52***
cf X mr	9	7.9	1.26
Residual	9	6.2	
Total	31		

*** $p < .001$

attention to single stimulus properties, stimulus generalization (measured in terms of total response probability) can be represented by an additive metric in accordance with the discrimination hypothesis.

Generalization gradients of R_1 latency and discrimination gradients of R_2 latency for Group I are shown in Fig. 5 for the amber light condition and in Fig. 6 for the blue light condition. These gradients result from pooling the data over stimulus presentations and over Ss for each light condition separately. Each data point represents the geometric mean latency of all R_1 or all R_2 responses evoked by a given stimulus. The number of separate latency measurements entering each mean determination varied (for the latencies actually plotted) from 9 to 50, depending on the number of R_1 or R_2 response evoked by each stimulus. Generalization latencies at the 900 cps level of cf under the amber light condition and at the 16 sps level of mr under the blue light condition were not plotted, because either no R_1 responses were evoked by the corresponding stimuli or too few responses were evoked to determine latency reliably. The same was true for discrimination (R_2) latencies at the 500 cps level of cf under the amber light condition and at the 2 sps level of mr under the blue light condition. Analyses of generalization and discrimination response latencies under the different light conditions were thus undertaken on reduced data tables resulting from the deletion, in each case, of the stimulus level indicated above.

It is evident in Figs. 5 and 6 that, in accordance with the findings for response frequency, gradients of response latency are steeper along the attended stimulus dimension under each light condition than along the unattended dimension. This result shows up particularly well for discrimination latencies. With the exception of one outlying observation (represented in the bottom graphs of Fig. 5 by the point corresponding to the

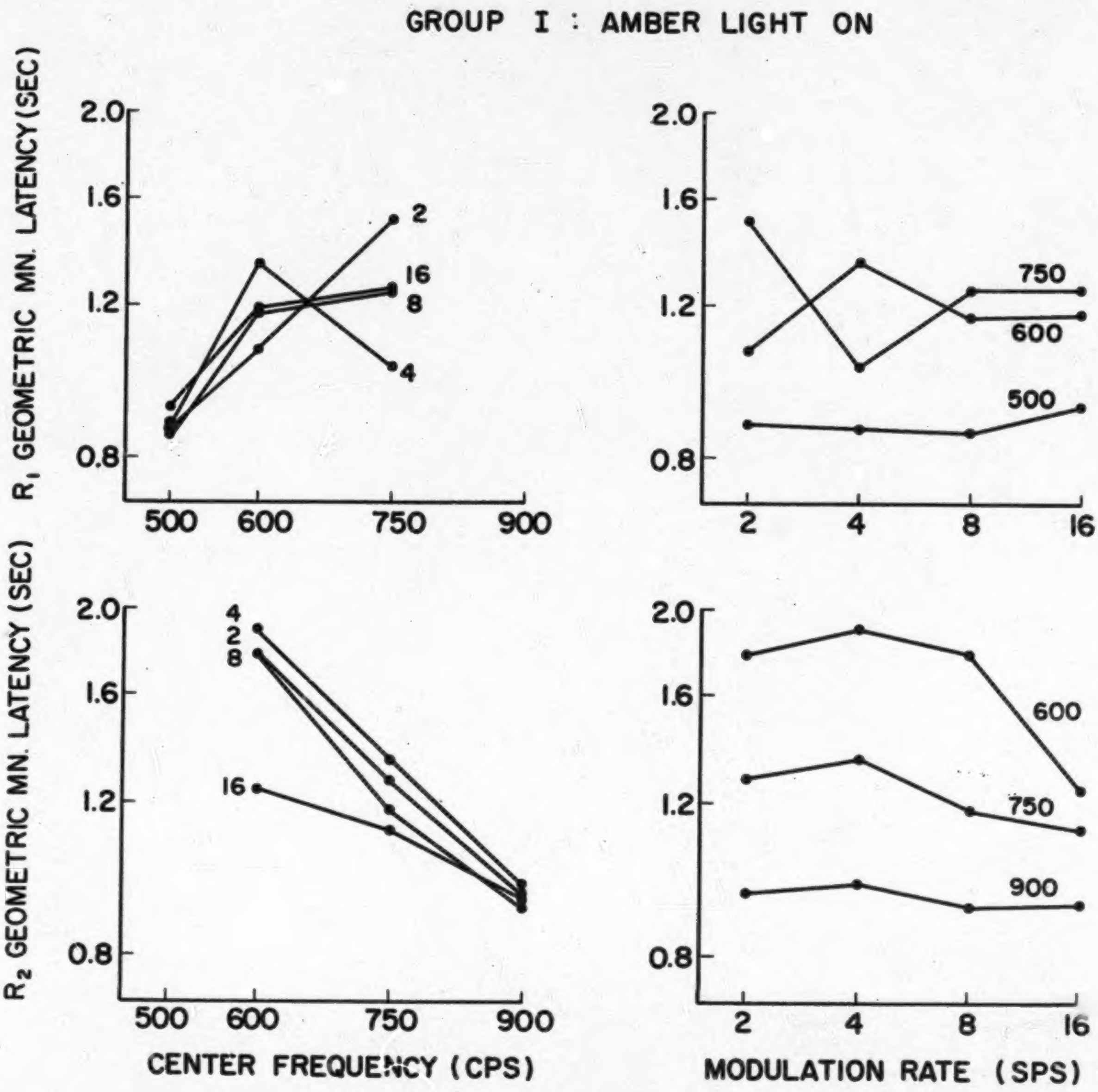

Fig. 5. Response latency under the amber light condition. Both R_1 latency (top graphs) and R_2 latency (bottom graphs) are shown as a function of cf, with mr as a parameter (left), and as a function of mr, with cf as a parameter (right). Each point represents the geometric mean latency of all R_1's (or R_2's) evoked in the course of 50 presentations of the corresponding stimulus (pooling over Ss). Ordinates are scaled in logarithmic units and labeled in linear units.

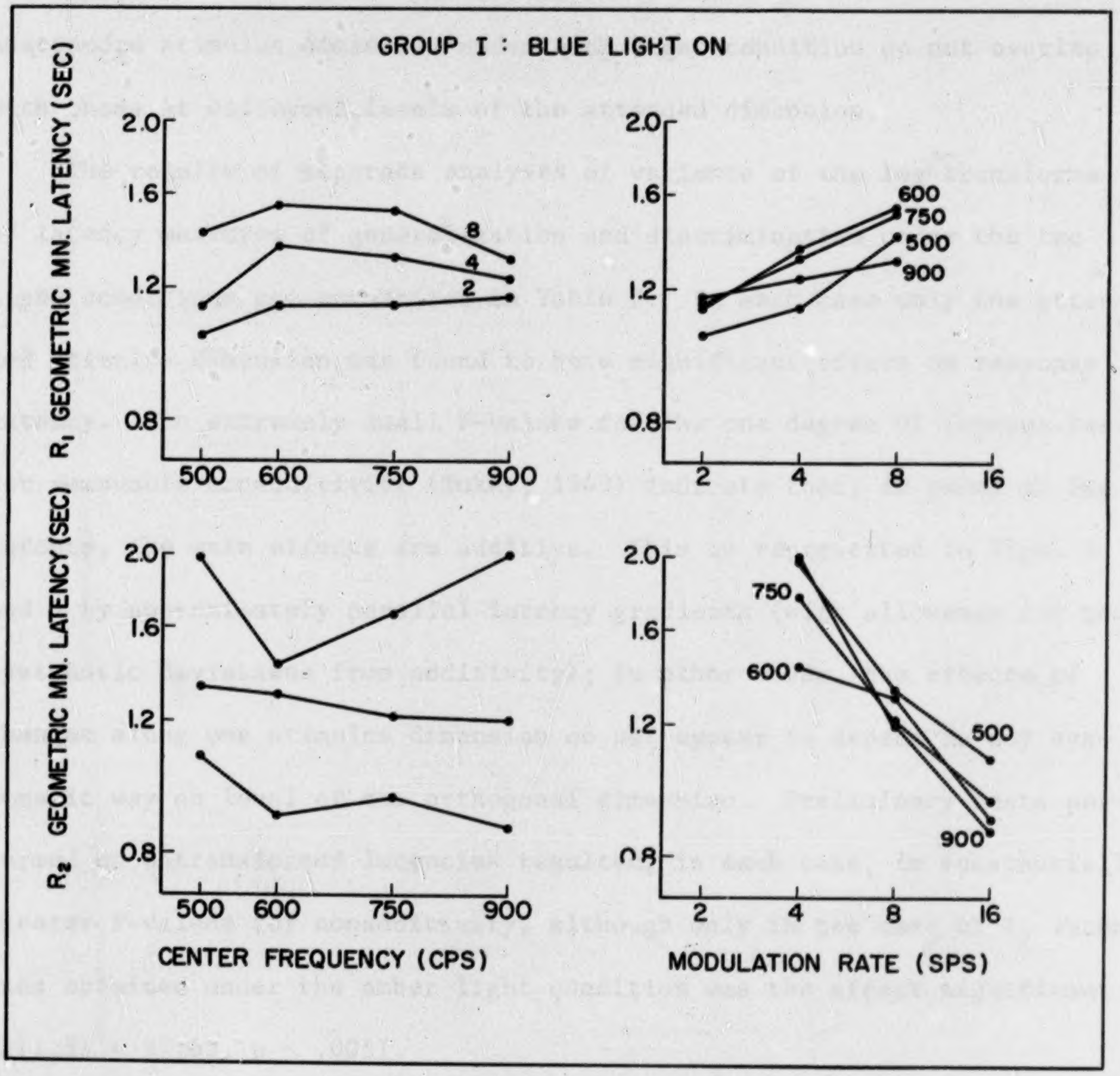

Fig. 6. Response latency under the blue light condition. Both R_1 latency (top graphs) and R_2 latency (bottom graphs) are shown as a function of cf, with mr as a parameter (left), and as a function of mr, with cf as a parameter (right). Each point represents the geometric mean latency of all R_1's (or R_2's) evoked in the course of 50 presentations of the corresponding stimulus (pooling over Ss). Ordinates are scaled in logarithmic units and labeled in linear units.

600 cps/16 sps stimulus) discrimination latencies at all levels of the unattended stimulus dimension under each light condition do not overlap with those at different levels of the attended dimension.

The results of separate analyses of variance of the log transforms of latency measures of generalization and discrimination under the two light conditions are summarized in Table 2. In each case only the attended stimulus dimension was found to have significant effect on response latency. The extremely small F-values for the one degree of freedom tests for removable nonadditivity (Tukey, 1949) indicate that, in terms of log latency, the main effects are additive. This is represented in Figs. 5 and 6 by approximately parallel latency gradients (with allowance for nonsystematic deviations from additivity); in other words, the effects of changes along one stimulus dimension do not appear to depend in any systematic way on level of the orthogonal dimension. Preliminary tests performed on untransformed latencies resulted, in each case, in substantially greater F-values for nonadditivity, although only in the case of R_2 latencies obtained under the amber light condition was the effect significant ($F(1,5) = 37.63$, $p < .005$).

It is indicated that discrimination latencies are more sensitive measures of stimulus change than are generalization latencies: the range of systematic variation in R_2 latency is almost double that for R_1 latency, in spite of the fact that the corresponding differences in response frequency are approximately equal. For example, under the blue-light condition the geometric mean R_1 latency (resulting from pooling across levels of cf) ranges from 112 csec (based on 200 responses) for the 2 sps level of mr to 144 csec (based on 63 responses) for the 8 sps level. In contrast, overall geometric mean R_2 latency ranges from 95 csec (based on 185 responses) for 16 sps to 167 csec (based on 53 responses) for 4 sps.

TABLE 2

ANALYSIS OF VARIANCE OF MEAN LOG RESPONSE LATENCY TO STIMULI VARYING IN CENTER FREQUENCY AND MODULATION RATE UNDER AMBER AND BLUE LIGHT CONDITIONS: GROUP I

A. GENERALIZATION (R_1) LATENCY

SOURCE	AMBER			BLUE		
	df	MS	F	df	MS	F
Center frequency	2	28,644	8.36*	3	1,889	3.50
Modulation rate	3	374	.12	2	12,283	22.81**
Residual	6	3,185		6	538	
Nonadditivity	(1)	502	.14	(1)	148	.23
Balance	(5)	3,722		(5)	616	

B. DISCRIMINATION (R_2) LATENCY

SOURCE	AMBER			BLUE		
	df	MS	F	df	MS	F
Center frequency	2	59,824	35.09***	3	3,819	4.48
Modulation rate	3	4,984	2.92	2	59,918	70.27***
Residual	6	1,705		6	853	
Nonadditivity	(1)	781	.41	(1)	215	.22
Balance	(5)	1,890		(5)	980	

* $p < .05$

** $p < .01$

*** $p < .001$

These comparisons represent a main effect difference of 32 csec for generalization latency and of 72 csec for discrimination latency. Although the differences pertain to different levels of mr, the stimulus changes involved are equal in log units. More to the point, however, as indicated above, both stimulus differences give rise to approximately the same difference in response frequency measured in relation to these levels.

The question concerning the form of the metric appropriate for measures of multidimensional stimulus control arising from the indiscriminate pooling of data over different occurrences of selective and exclusive control by single stimulus properties is served by these data by pooling experimental observations across light conditions. It was already shown by the absence of cf × mr interaction effects in the analysis of variance of generalization (R_1) probabilities that a simple additive metric is appropriate for this measure of stimulus control. This result is shown for the complementary measures of discrimination (R_2) probability in Fig. 7(a) in the form of a scatter plot, relating obtained $P(R_2)$ for each test stimulus to fitted values $\hat{P}(R_2)$ derived from the analysis of variance assumption of simple structure, that is, the assumption that each cell (stimulus) effect represents the simple sum of the corresponding row and column effects. The linear correlation between obtained and fitted values is .993 which means that simple structure accounts for 98.6% of the total variation in obtained measures of discrimination probability.

The pooling of R_1 latencies across Ss, stimulus presentations and light conditions results in measures of generalization latency that show relatively little systematic covariation with stimulus change. This is indicated in the scatter plot shown in Fig. 7(b) which relates mean log R_1 latency for each of the 16 test stimuli to the corresponding measure of R_1 probability. The coefficient of linear correlation between these

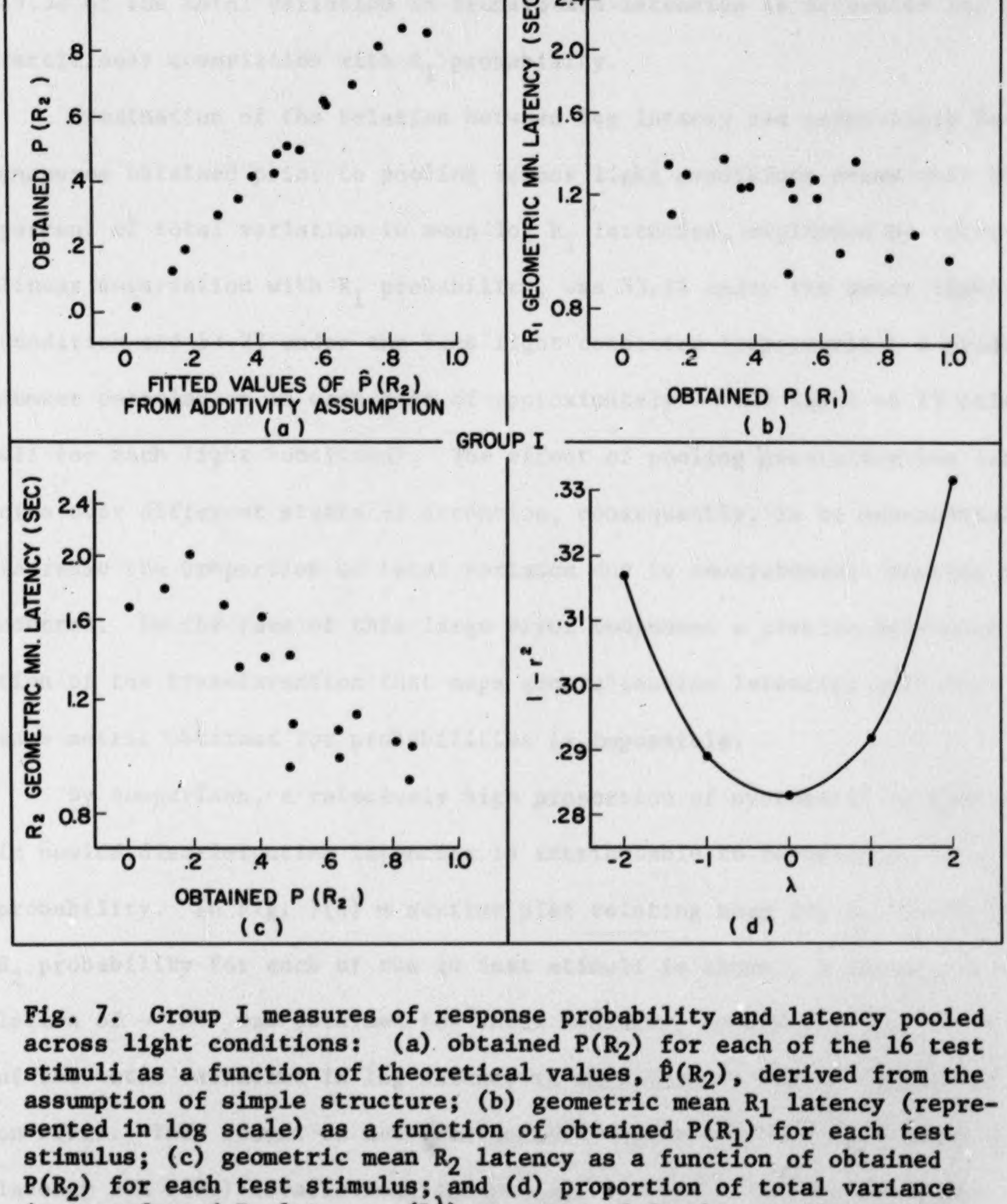

Fig. 7. Group I measures of response probability and latency pooled across light conditions: (a) obtained $P(R_2)$ for each of the 16 test stimuli as a function of theoretical values, $\hat{P}(R_2)$, derived from the assumption of simple structure; (b) geometric mean R_1 latency (represented in log scale) as a function of obtained $P(R_1)$ for each test stimulus; (c) geometric mean R_2 latency as a function of obtained $P(R_2)$ for each test stimulus; and (d) proportion of total variance in transformed R_2 latency (L^{λ}) unaccounted for by rectilinear covariation with $P(R_2)$, as a function of the parameter λ.

alternative measures of stimulus control is —.541. In other words, only 29.3% of the total variation in transformed latencies is accounted for by rectilinear covariation with R_1 probability.

Examination of the relation between log latency and probability for measures obtained prior to pooling across light conditions shows that the percent of total variation in mean log R_1 latencies, explained by rectilinear covariation with R_1 probability, was 53.1% under the amber light condition and 53.2% under the blue light condition (representing a product moment correlation in each case of approximately —.729 based on 15 stimuli for each light condition). The effect of pooling generalization latencies over different states of attention, consequently, is to substantially increase the proportion of total variance due to nonsystematic sources of control. In the face of this large error component a precise determination of the transformation that maps generalization latencies into the same metric obtained for probabilities is impossible.

By comparison, a relatively high proportion of systematic variation in pooled discrimination latencies is attributable to covariation with R_2 probability. In Fig. 7(c) a scatter plot relating mean log R_2 latency to R_2 probability for each of the 16 test stimuli is shown. A linear correlation of —.847 was obtained for these measures, indicating that 71.7% of the total variation in log latency is explained by linear regression on $P(R_2)$. This result is not substantially different from that found for latency and $P(R_2)$ covariation, determined separately for the two light conditions prior to pooling. A product moment correlation (based on response measures for all 16 test stimuli) of —.879 was obtained between mean log latency and R_2 frequency under the amber light condition and of —.868 (for 12 stimuli since no R_2 emissions occurred to 4 of the 16 test stimuli) under the blue light condition. It is notable that the degree of linear relation between discrimination latency and probability is practically identical for the two light conditions, as was the case for the corresponding generalization measures.

Examination of the residuals about the line of regression of pooled mean log latency on $P(R_2)$ revealed neither a systematic pattern of dependence on level of $P(R_2)$ nor any detectable evidence for the presence of a curvilinear relationship. This is partial evidence for the appropriateness of the chosen mode of expression for obtained latency measures.

A more systematic analysis was undertaken in which linear correlations between $P(R_2)$ and alternative transformations of response latency were separately determined in order to find a mode of expression that maximizes the proportion of variance in transformed latencies explained by linear regression on $P(R_2)$. Attention was restricted to the single parameter family of power transformations, resulting from replacing observed L by L^λ and determining its correlation with $P(R_2)$ for various values of λ (Anscombe & Tukey, 1963; Box and Cox, 1964). The proportion of total variation in L^λ unexplained by linear regression on $P(R_2)$ is plotted in Fig. 7(d) as a function of λ ($\lambda = 0$ is to be interpreted as the logarithmic transformation). The smooth curve fitted to computed values of $1-r^2$ for various L^λ and $P(R_2)$ indicates that the component of residual (unexplained) variance is minimal when response latencies are scaled in log units. In this sense, relative to the family of transformations considered, a log transformation may be considered to be the most appropriate choice of scale for discrimination latencies. This outcome is only suggested by the analysis and can hardly be considered conclusive. In fact, both reciprocal latency and latency expressed in linear scale enter nearly as close a relation with $P(R_2)$ as does log latency.

The large nonsystematic variation in latency measures of stimulus control is not surprising, in view of the fact that these measures were based on pooled observations obtained under conditions of selective attention to different stimulus properties by different Ss and over repeated stimulus presentations. At the level the foregoing analyses were carried out these sources of experimental error were not analyzed. Each mean latency determination was treated as a single observation and given equal weight in the analysis in spite of the fact that the number of responses entering each determination differed greatly between stimuli depending on the number of R_1 or R_2 responses evoked. The stimulus effects (defined in terms of mean latency) were evaluated by comparison with estimates of measurement error based on the residual variation remaining after fitting an appropriate linear model to the data.

The variability of individual latency determinations was quite large in relation to overall effects measured by mean latencies. For example, pooling across Ss and all test presentations of the 500 cps/2 sps tone (which served as S^D for R_1 during training), regardless of light condition, a total of 98 R_1 responses were emitted. The latencies of these responses were distributed over a range of 43 to 292 csec and about a mean of 103.3 csec with a standard deviation of 44.0 csec. The distribution was positively skew with an index of skewness, β_1 (calculated from the moments), equal to + 1.649. The individual R_2 latency determinations for the 85 responses evoked by the 900 cps/16 sps tone were distributed over a range of 47 to 292 csec about a mean of 98.6 csec with a standard deviation of 45.3 csec. This distribution was also positively skew with β_1 = + 1.637. These latencies pertain to the stimulus that served to maximally evoke R_2 during testing. Thus, for the two stimuli that most effectively controlled R_1 and R_2 responding during testing, with minimum overall latency of responding, the distributions of generalization and discrimination latencies were very similar. The choice of geometric mean latency, as a measure of stimulus control, rather than the arithmetic mean of each sample, was based on the presence of positive skewness.

Estimates of standard error of the mean based on these specific samples of R_1 and R_2 latency was 4.5 and 4.9 csec, respectively. These estimates compare with root mean square error measures obtained in analyses of variance of geometric mean R_1 and R_2 latencies to all stimuli of 6.4 and 5.7 csec, respectively.

A large component of variation in both generalization and discrimination latencies can be attributed to differences between Ss. This source of variation in generalization latencies is highly significant compared with pooled within Ss variation [$F(9, 88) = 5.06$, $p < .001$]. Between S differences in discrimination latencies is also highly significant [$F(9, 75) = 5.50$, $p < .001$].

Generalization testing: Group II. Test conditions for the 10 Ss in Group II were identical to those for Group I with one important exception. Rather than controlling selective attention to single stimulus properties throughout generalization testing, the last 6 sequences of test stimuli were presented with the amber and blue lights simultaneously illuminated. Thus, after training and initial testing with operations insuring selective attention to single stimulus properties, the visual cues were compounded (an operation formally analogous to verbally instructing S to attend both discriminative aspects of the stimulus).

The results of generalization testing with the amber or blue light on alone were essentially in accord with the previous findings. Ss attended cf differences only when the amber light was on and mr differences only when the blue light was on. Although based on only two presentations of each test stimulus to each S, measures of response latency, for both R_1 and R_2, also showed that differential control was exerted only by the attended stimulus property under each light condition. The correlation between generalization latency (pooled across light conditions) and R_1 frequency was on the order of $-.45$. The correlation between the latency of a discrimination (also pooled across light conditions) and R_2 frequency was $-.70$. Thus, in accordance with Group I results, the latency and probability of a discrimination show a greater degree of systematic covariation than do the corresponding measures of generalization.

Figure 8 shows the results of testing with both the amber and blue lights on simultaneously. The bottom graphs show the relative frequency of R_2 responding to each of the 16 combinations of cf and mr. The left graph shows percent discrimination to stimuli varying in cf (with levels of mr as the parameter) and the right graph shows the same measures as a function of mr (with levels of cf as the parameter). It is apparent that both dimensions exercise behavioral control, although changes in cf appear to be more effective overall than changes in mr. The gradients appear to be reasonably parallel, that is, the effects of changes in one dimension are approximately constant across levels of the second dimension. This indicates that the effects of stimulus changes, as measured by response frequency, can be represented by additive row and column constants. A scatter plot relating obtained $P(R_2)$ for each test stimulus to fitted values $\hat{P}(R_2)$ derived from the assumption of simple additive structure is shown in Fig. 9(a). The linear correlation between obtained and fitted values is .994; in other words, simple structure accounts for 98.9% of the total variation in obtained measures of discrimination probability. This outcome is practically identical to that obtained previously under conditions of controlled attention to single stimulus properties. (cf. Fig. 7(a).)

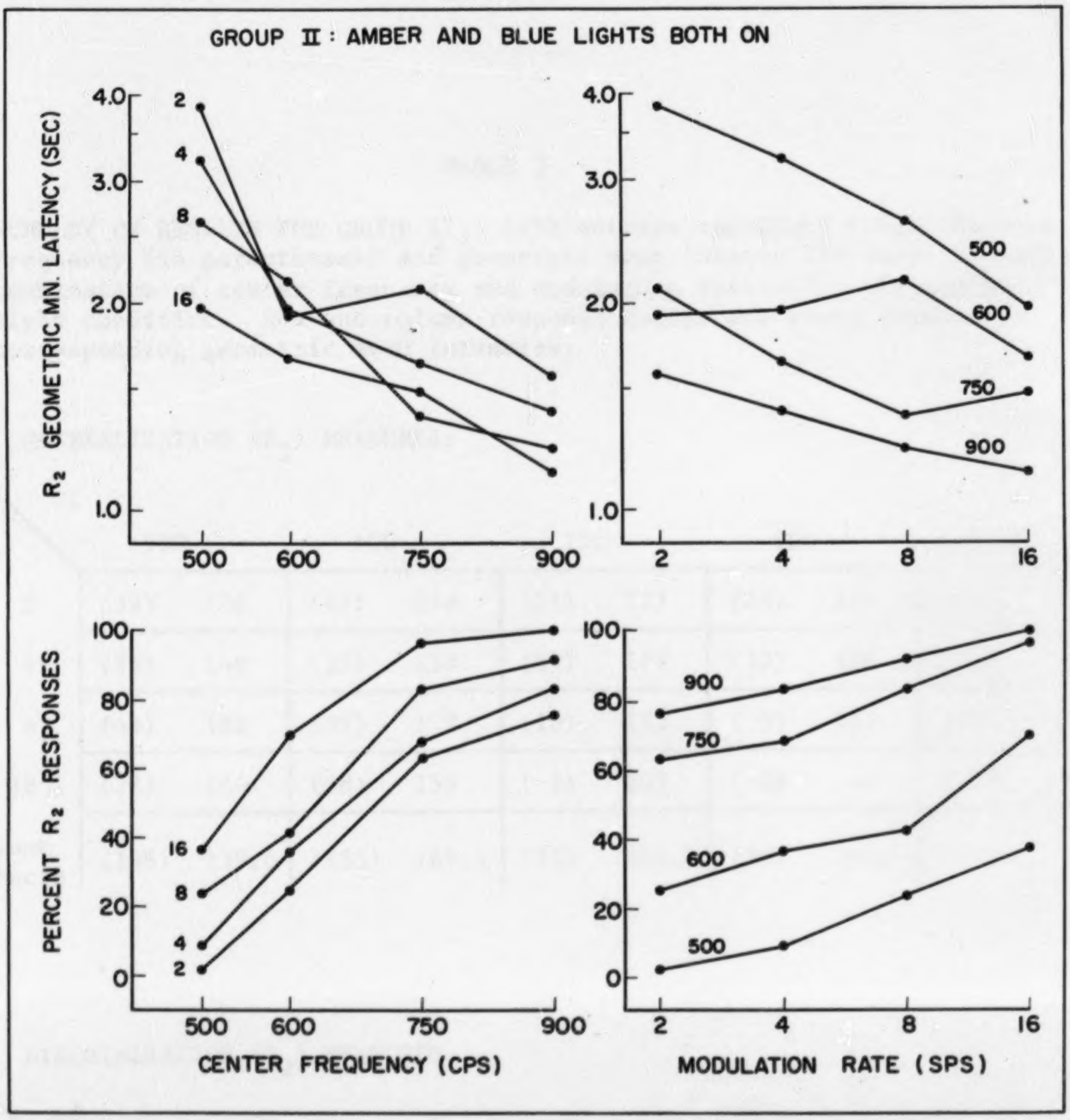

Fig. 8. Response probability and latency under the combined light condition: Group II. Geometric mean R_2 latency is shown (top graphs) as a function of cf, with mr as a parameter (left), and as a function of mr, with cf as a parameter (right). Bottom graphs show corresponding measures of R_2 probability as a function of cf, with mr as a parameter (left) and as a function of mr, with cf as a parameter (right). Each point is based on the total number of R_2 responses emitted in the course of 60 presentations of each stimulus (pooling over Ss).

TABLE 3

SUMMARY OF RESULTS FOR GROUP II. Cell entries represent total response frequency (in parentheses) and geometric mean latency (in csec) to each combination of center frequency and modulation rate under the combined light condition. Row and column response totals are given along with corresponding geometric mean latencies.

A. GENERALIZATION (R_1) MEASURES:

mr \ cf	500		600		750		900		row effects	
2	(59)	126	(45)	178	(21)	177	(14)	214	(139)	156.5
4	(54)	148	(37)	158	(19)	174	(10)	190	(120)	158.0
8	(45)	152	(35)	177	(10)	173	(5)	151	(95)	163.0
16	(38)	144	(18)	159	(2)	207	(0)	---	(58)	150.2
column effects	(196)	139.6	(135)	169.5	(52)	176.3	(29)	193.5		

B. DISCRIMINATION (R_2) MEASURES:

mr \ cf	500		600		750		900		row effects	
2	(1)	385	(15)	191	(36)	209	(44)	156	(96)	181.6
4	(5)	322	(21)	194	(41)	163	(50)	139	(117)	161.6
8	(14)	263	(25)	214	(50)	136	(55)	122	(144)	150.7
16	(22)	196	(42)	166	(58)	148	(59)	113	(181)	144.1
column effects	(42)	233.0	(103)	186.0	(185)	158.2	(208)	129.8		

Geometric mean R_2 latencies are plotted in log scale in the top graphs of Fig. 8. The number of responses entering each latency determination varies from 1 to 59, depending on the number of R_2 emissions to each stimulus. In spite of the small n involved in some of these determinations, discrimination latency appears to be systematically covariant with stimulus change along both dimensions. A one degree of freedom test for removable nonadditivity (Tukey, 1949) indicates the existence of a significant linear × linear component of cf × mr interaction for effects measured in linear units (centiseconds) of discrimination latency [$F(1, 8) = 12.98$, $p < .01$]. Conversion to logarithmic scale, however, renders main effects essentially additive; the corresponding F-value for nonadditivity is reduced to an insignificant level [$F(1, 8) = 2.10$, $p < .20$]. In log scale, both cf effects and mr effects are significant; $F(3, 9) = 24.58$, $p < .001$, and $F(3,9) = 5.81$, $p < .05$, respectively.

Geometric mean R_1 latencies and geometric mean R_2 latencies for all 16 test stimuli are presented in Table 3, along with the number of responses entering each determination. Geometric mean response latency at each level of cf (pooling over levels of mr) and at each level of mr (pooling over levels of cf) is shown in each row and column margin. In accordance with the findings for Group I, discrimination latency appears to be a much more sensitive measure of stimulus change than generalization latency. The change in discrimination latency from 233.0 csec (based on 42 responses) to 129.8 csec (based on 208 responses) for the 500 cps and 900 cps levels of cf, respectively, represents a differential effect of 103.6 csec. In contrast, the corresponding generalization latencies are 139.6 csec (based on 196 responses) and 193.5 csec (based on 29 responses), representing a differential effect of only 53.9 csec. Although these comparisons correspond to almost identical differences in response frequency,

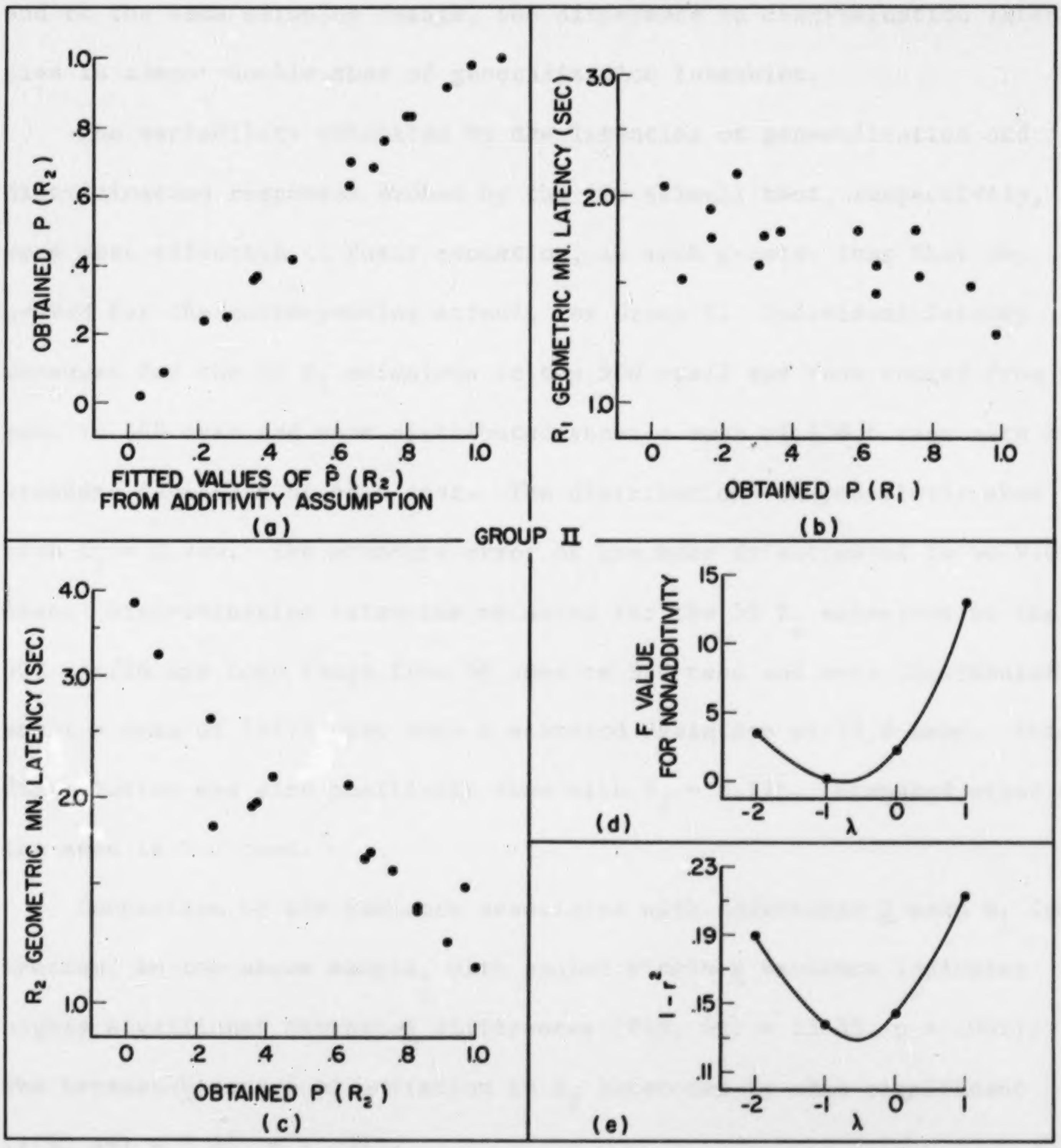

Fig. 9. Group II measures of response probability and latency: (a) obtained $P(R_2)$ for each of the 16 test stimuli as a function of theoretical values, $\hat{P}(R_2)$, derived from the assumption of simple structure; (b) geometric mean R_1 latency (represented in log scale) as a function of obtained $P(R_2)$ for each test stimulus; (c) geometric mean R_2 latency as a function of obtained $P(R_2)$ for each stimulus; (d) F-value for the one degree of freedom test for nonadditivity performed on transformed R_2 latency (L^{λ}) as a function of the parameter λ; and (e) proportion of total variance in transformed R_2 latency (L^{λ}) unaccounted for by rectilinear covariation with $P(R_2)$, as a function of the parameter λ.

and to the same stimulus change, the difference in discrimination latencies is almost double that of generalization latencies.

The variability exhibited by the latencies of generalization and discrimination responses evoked by the two stimuli that, respectively, were most effective in their evocation, is much greater than that observed for the corresponding stimuli for Group I. Individual latency measures for the 59 R_1 emissions to the 500 cps/2 sps tone ranged from 66 csec to 368 csec and were distributed about a mean of 138.6 csec with a standard deviation of 68.4 csec. The distribution was positively skew with $\beta_1 = 1.740$. The standard error of the mean is estimated to be 9.0 csec. Discrimination latencies measured for the 59 R_2 emissions to the 900 cps/16 sps tone range from 56 csec to 328 csec and were distributed about a mean of 127.2 csec with a standard deviation of 74.0 csec. This distribution was also positively skew with $\beta_1 = 2.175$. Standard error of the mean is 9.7 csec.

Comparison of the variance associated with individual S̲ mean R_1 latencies, in the above sample, with pooled within-S̲ variance indicates highly significant between-S̲ differences [$F(9, 49) = 15.85$, $p < .001$]. The between-S̲ source of variation in R_2 latencies is also significant [$F(9, 49) = 3.02$, $p < .01$].

In spite of the variability in individual latencies and the small number of responses involved in the estimate of response probability, a relatively high degree of systematic covariation is indicated between measures of stimulus control, based on geometric mean latencies (pooling across stimulus presentations and S̲s) and corresponding measures of response probability. In Fig. 9(b) a scatter plot is shown, relating mean log R_1 latency and obtained $P(R_1)$ for 15 test stimuli (no R_1 was emitted to one stimulus). Although only 44.9 % of the total variation in generalization latency is explained by rectilinear covariation with $P(R_1)$ (as indicated by a product moment correlation of $-.667$), examination of the scatter plot reveals no detectable curvilinearity. The covariation between discrimination latency and obtained $P(R_2)$ is shown, in scatter plot form, in Fig. 9(c). A product moment correlation of $-.925$ indicates a very close relation between log latency and $P(R_2)$; approximately 85.6% of the total variation in latency is explained by linear regression on $P(R_2)$.

Figure 9(e) shows the proportion of total variance in transformed latencies (L^{λ}) unexplained by linear regression on $P(R_2)$ for various values of λ (recalling that $\lambda = 0$ is interpreted as the logarithmic transformation). The smooth curve fitted by eye to the calculated values indicates that residual variance may be minimized by a transformation intermediate to the reciprocal and logarithmic transformations, although this minimum represents a negligible gain in variance accounted for over that associated with either the reciprocal or logarithmic mode of expression (the correlation between $L^{-1/2}$ and $P(R_2)$ is .932).

An alternative approach to finding a mode of expression for latency in terms of which a succinct metric representation may be obtained is provided by finding the transformation that minimizes the F-value for the one degree of freedom test for nonadditivity (Anscombe & Tukey, 1963). This procedure has an advantage over the preceding analysis which depends upon the assumption that response probabilities have simple structure, but its utility is limited when applied to data tables having rows or columns that are not identically ordered. The test is sensitive to only one source of nonadditivity, namely, that associated with the linear × linear component of interaction which is, in principle, removable by rescaling the dependent variable. It measures the extent to which the size of the interaction effect for each treatment combination depends upon level of the corresponding main effects. One or two outlying observations, causing row or column effects to be inconsistently ordered, as in the present data, may, depending on where they occur, appear to constitute a measurable amount of removable nonadditivity and hence bias the outcome.

The tests were performed on discrimination latencies for the linear, logarithmic, reciprocal, and reciprocal squared transformations ($\lambda = 1, 0, -1$, and -2, respectively). The corresponding F-values for nonadditivity are shown in Fig. 9(d). The smooth curve fitted to the calculated values appears to reach a minimum in the neighborhood of $\lambda = -1$ (the reciprocal transformation). For the F-values plotted, however, only the one obtained for latencies expressed in linear scale indicates significant nonadditivity. For 1 and 8 degrees of freedom the F-value corresponding to the 0.05 significance level is 5.3. Hence, we may presume the results are in substantial agreement with those of the correlation analysis.

Discussion

The results of this study are consistent with the notion that the metric structure of behavioral data arises from the indiscriminate pooling of data across instances of selective and exclusive attention to single stimulus properties. A simple additive structure was obtained for response probabilities, both under conditions in which Ss were cued to selectively attend stimulus changes along one dimension at a time, as well as under conditions that presumably set the occasion for simultaneous control by both dimensions of stimulus variation. The major difference in the results obtained under the two conditions is not in the metric structure of the data but in the relative importance of the two stimulus dimensions in the apparent control of differential responding. For the pooled data of Group I the main effects of cf and mr variation are approximately equal, as measured in terms of both differences in relative response frequency and differences in discrimination latency. This outcome is due to the fact that selective attention was under visual control, and the experimental design insured that attention to the separate dimensions occurred equally often. For Group II, however, despite the concomitance of visual cues that, separately, were effective in controlling attention to one or the other dimension of stimulus change, behavior was neither under control of both dimensions simultaneously nor was behavior equally controlled by both (see Table 3 and Fig. 8). Changes in cf appear to have exerted a much greater degree of control than mr changes, as measured in terms of both pooled response frequency and latency. However, both cf effects and mr effects were smaller under the combined light condition than under the condition in which S̲s were cued to attend one or the other dimension exclusively. These facts, together with the additive structure of the data indicate that the dimensions were selectively attended and that over stimulus presentations and S̲s the dimension of cf variation was attended more often than that of mr.

The restriction of reinforcement during training, to differential responding in the presence of cf or mr differences exclusively, did not preclude the possibility of cf × mr interaction effects in the presence of concomitant variation or of the emergence of effective dimensions of control that cut across the physical dimensions of the space. In view of the results, however, it may be inferred that the effective dimensions of control were orthogonal and parallel to the physical dimensions, and that (for the range or values employed) the physical properties in question do not interact in the control of discriminative behavior.

This study represents the only attempt to date to measure two-dimensional stimulus differences in terms of disjunctive response times, despite the long history of response latency as a dependent variable in investigations of stimulus control. Woodworth (1914) credits Cattell for proposing, as early as 1887, that the psychological difference between two stimuli can be measured by the time of reaction to that difference. Numerous psychophysical studies involving stimuli that vary along single dimensions have shown that, in general, the latency of a 'same' response increases as the difference between a standard and comparison stimulus increases. It has also been found with two and three categories of judgment that, on the average, the latency of the stochastically dominant response (one with greatest probability of emission) is shorter than the latency of a non-dominant response. On the other hand, the correlative variation between judgment time and stimulus difference has invariably been found to be poorer than that between response-frequency and stimulus difference, and several studies have failed to find a significant correlation at all. Exceptions to the other finding have also been reported. In the face of these inconsistencies psychophysical investigators have generally rejected the method of judgment time in preference for other psychophysical methods that yield more consistent results.

In a comprehensive review of stimulus generalization research Mednick & Freedman (1960) cite 18 experiments in which generalization has been measured in terms of response latency. Orderly generalization latency-gradients have been found in some cases but several investigators have failed to find any systematic relation between response latency and stimulus change, despite evidence of stimulus control from concomitant measures of response frequency. Despite the prominent status of latency as a measure of response strength in behavior theory (Hull, 1943) and the recent proliferation of mathematical models proposed to characterize temporal properties of choice behavior (e.g. Luce, 1959; Audley, 1960; Stone, 1960; LaBerge, 1962; Kintsch, 1963), because of the variability inherent in response time and its sensitivity to factors outside the range of experimental control, few investigations have been successful in establishing precise mathematically expressed relations between response latency and other variables.

The present findings exhibit very little systematic covariation between generalization latency and stimulus change. Discrimination latency, however, appears to be a highly sensitive measure of stimulus control. As shown in Fig. 8, reasonably orderly discrimination latency gradients were obtained along both dimensions of stimulus variation. A difference between generalization and discrimination latencies with regard to their degree of correlative variation with response probability was observed for both experimental groups (compare Figs. 7(b) and 7(c) with Figs. 9(b) and 9(c)). Aside from the fact that the visual conditions differed for the two groups, in other respects the two sets of findings may be viewed as independent replications of the empirical relations under investigation. In both sets of data the overall effect of stimulus change, as measured by discrimination latency, is approximately twice as great as that observed for generalization latency. This contrast is consistent with Landahl's conjecture that two separate neural mechanisms are involved in the perception of similarities and the perception of differences. In this connection, generalization (perception of similarities) is viewed as being controlled directly by the common elements of stimuli, whereas discrimination (perception of differences) is mediated by some kind of 'subtractive' neural mechanism (Landahl, 1945).

In the choice of a mode of expression for response latency in terms of which simplicity of metric structure is achieved, the evidence from this study points to a relatively restricted set of alternative transformations of which the most reasonable candidates are the reciprocal and the logarithmic transformations. The reciprocal transformation has a natural appeal for the analysis of response latencies because it is open to the simple interpretation that it is the speed of the response that is to be considered. On the other hand, the logarithmic transformation is equally appealing because it leads to the interpretation that equal differences in discrimination probability correspond to equal ratios of discrimination latency; in other words, an additive structure for stimulus control defined in terms of probability implies a multiplicative structure in terms of response latency. The data permit only approximate inference about a choice of scale for latencies because the stimulus effects are not consistently ordered with respect to the two dimensions of stimulus variation. However, in view of the fact that these measures arise from the indiscriminate pooling of individual response times in the face of significant between subject differences and under conditions of selective attention to single stimulus properties this outcome is less surprising than is the fact that the latency functions appear to be reasonably orderly (exhibiting definite trend), and that a high degree of correlative variation is found to exist between pooled latency and relative response frequency.

The significant feature of this study is the finding that a metric treatment of stimulus control is possible, not despite the unreliability of the controlling relations that obtain between aspects of a stimulus and a response but, rather, because of it. It is because attention is selective and labile that averaged data exhibits additive structure.

CHAPTER IV

Summary and Conclusions

This thesis is concerned with the general problem of how the total effect of a multidimensional stimulus is compounded from the simple effects of its separate components in the context of stimulus generalization.

On the theoretical level the problem is studied in terms of the geometry of Minkowski. Stimuli are identified as points in Minkowski space, and the metric for the space is determined by the shape of the equal-level surfaces corresponding to prescribed amounts of generalization decrement. A special class of such spaces is characterized by the single parameter family of distance functions, known as the Minkowski r-metrics. Three models of stimulus generalization are represented by the Minkowski r-metric as r takes on the values 1, 2 and ∞, respectively. These three models represent the generalization decrement produced by a multidimensional stimulus as 1) a simple sum of concurrent, unidimensional discriminations, 2) a direct function of Euclidean distance and 3) a function only of the component stimulus change which, alone, is most readily discriminated. Intermediate values of r are, of course, admissible and may correspond to other hypotheses of stimulus generalization, or may be interpreted, simply, as a parameter of stimulus control that indicates how component stimulus effects are differentially weighted in predicting the total effect of a multidimensional stimulus.

A review of the literature reveals that existing experimental findings tend to converge on the general appropriateness of the $r = 1$ model. In each case, however, it was necessary to transform the given generalization measures in order to achieve this simple metric solution. This gives rise to the problem of identifying the response measure in terms of which a simple additive metric would apply directly.

It is shown on the basis of an a priori analysis of the controlling relations that can obtain between the component properties of a compound stimulus and a simple discriminative response that, when generalization is measured in terms of response probability, the appropriateness of the $r = 1$ model implies that behavior is under exclusive control of one or another dimension upon any one stimulus presentation and never under simultaneous control of two or more dimensions. This conclusion is contradictive of the contention that the triangle inequality property of distance cannot be satisfied by averaged data collected under conditions of selective and shifting attention to single stimulus dimensions and that, under these conditions, a spatial (metric) representation of the stimuli would be impossible.

In order to bring empirical evidence to bear on this issue the following experiment was carried out. Twenty human subjects were trained on a successive discrimination involving three frequency modulated sinusoidal tones that differed in center frequency and/or the rate of modulation. The pattern of reinforcement for the emission of either of two mutually exclusive responses (left or right key press) was contingent on the presence of one of two visual cues (an amber or a blue light). In this way, the stimulus property attended was brought under experimental control. The structure induced on the extended stimulus set was examined using generalization testing procedures with the two visual cues presented singly (to control selective attention) for 10 subjects, and with the cues presented in combination (setting the occasion for simultaneous control by both stimulus dimensions) for the other 10 subjects. Measures of both response probability and response latency were obtained.

A simple additive structure was obtained for response probabilities both under conditions in which subjects were cued to selectively attend stimulus changes along one dimension at a time as well as under conditions that were presumed to set the occasion for simultaneous control by both dimensions. The evidence indicated that, in the latter case, subjects were also selectively attending the separate stimulus dimensions.

Measures of discrimination latency were found to be closely related to corresponding measures of response probability, and, thus, were found covary systematically with stimulus change, but generalization latencies exhibited little systematic trend. In the choice of a mode of expression for response latency in terms of which simplicity of metric structure may be achieved the evidence from this study points to a relatively restricted set of alternative transformations of which the most reasonable candidates are the reciprocal and the logarithmic transformations.

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